Lecture 8: Multi-armed Bandits

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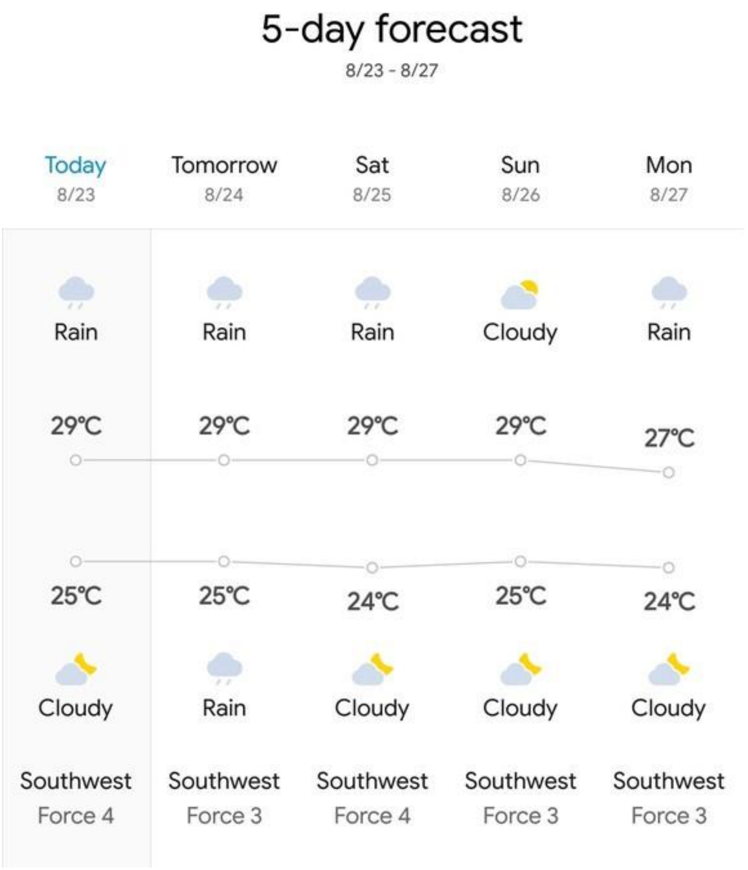
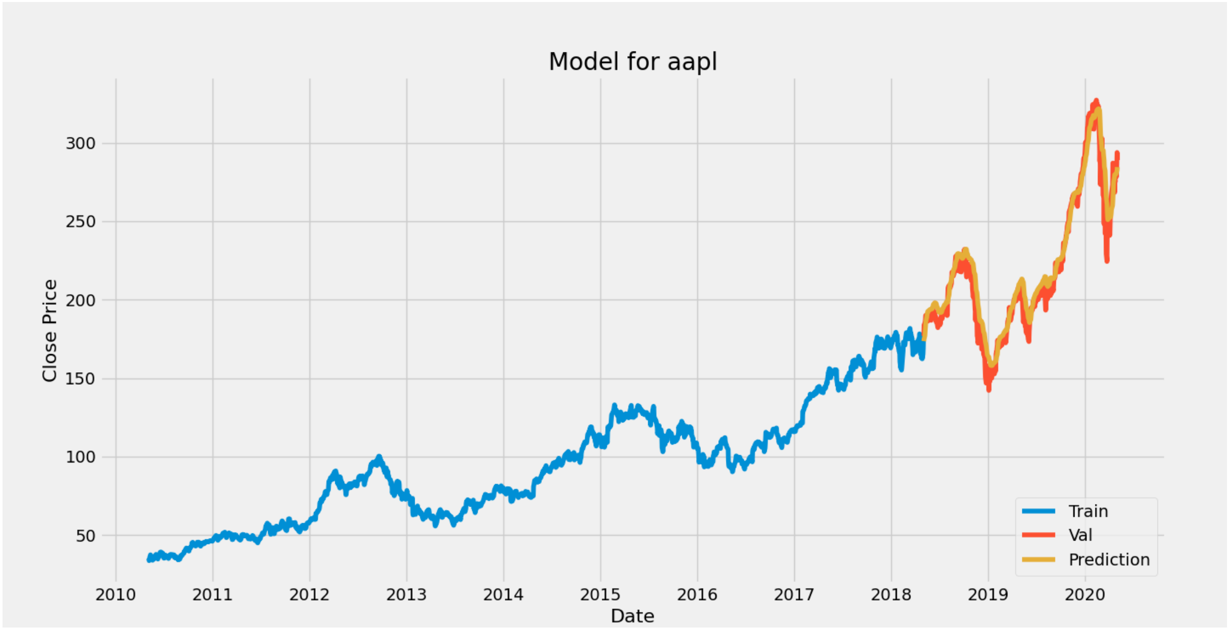
Online Learning w/ Full Information

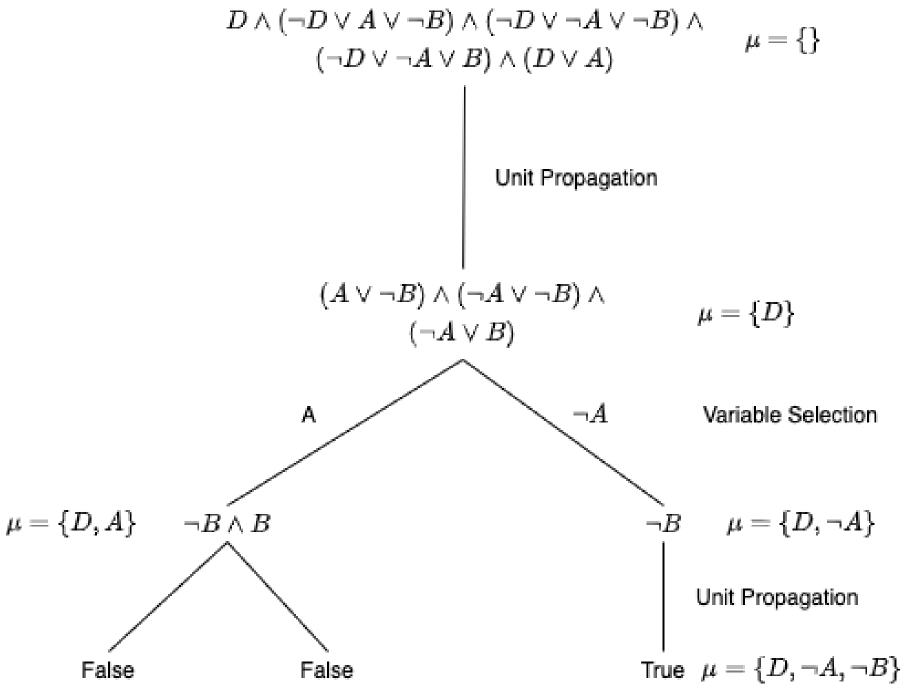
- Can observe feedback of every action





Online Learning w/ Bandit Feedback

- Can only observe feedback for the selected action



Bandits

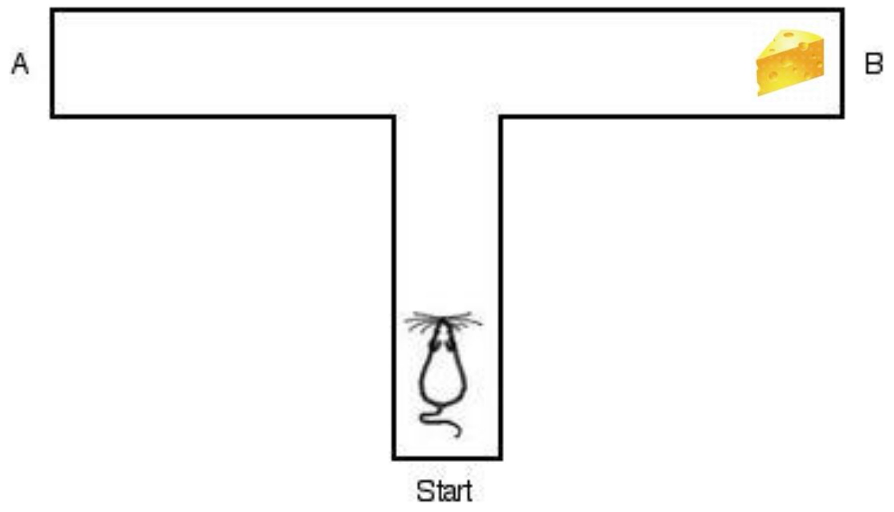
Time	1	2	3	4	5	6	7	8	9	10	11	12
Left arm	\$1	\$0			\$1	\$1	\$0					
Right arm			\$1	\$0								

- Five rounds to go. Which arm would you choose next?

What are bandits, and why should you care

What's in the name?

- First bandit algorithm proposed by Thompson (1933)



- Bush and Mosteller (1953) were interested in how mice behaved in a T-maze

Why care about bandits?

- Many applications
- They isolate an important component of reinforcement learning: exploration-vs-exploitation
- Theoretically guaranteed algorithms
- Rich and beautiful mathematics

Applications: Recommendation systems

- Yahoo news [Li et al. (2010)]

Applications: A part of RL

- A way of isolating an interesting part of reinforcement learning
  - Recommending items [Hu et al. (2018)]
  - Achieved more than 30% growth in GMV

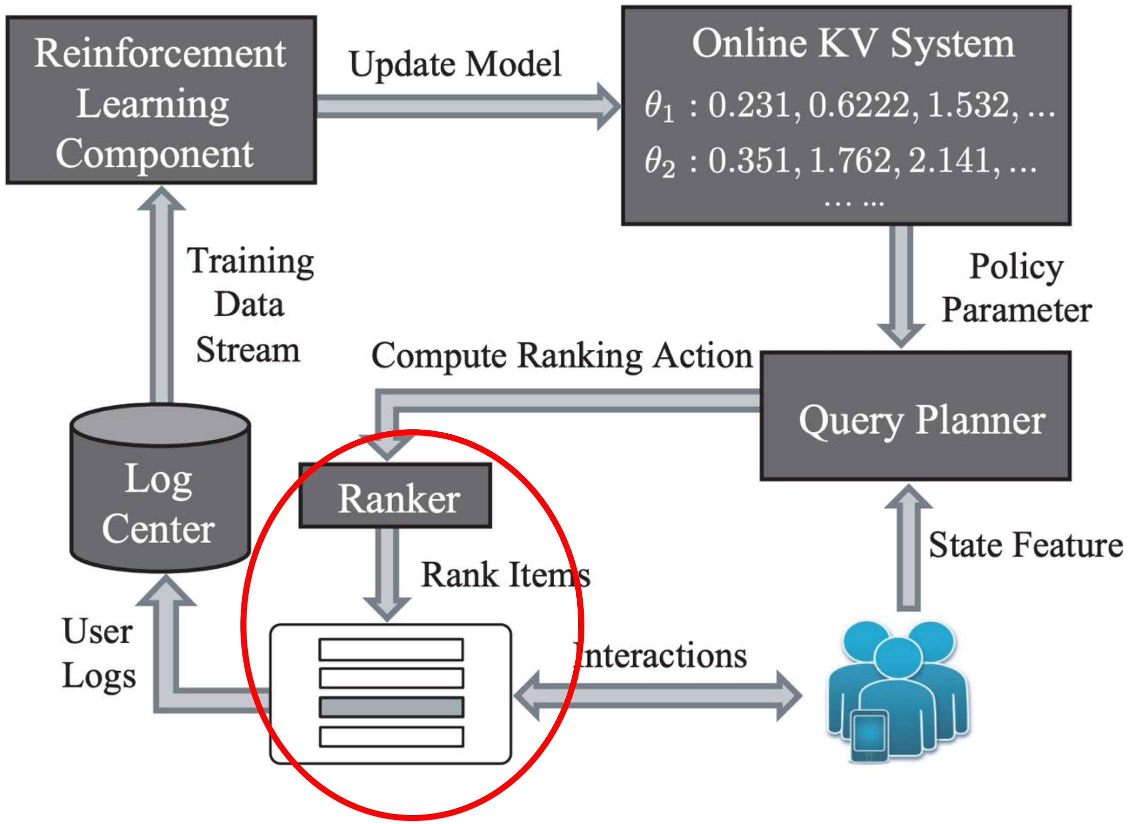
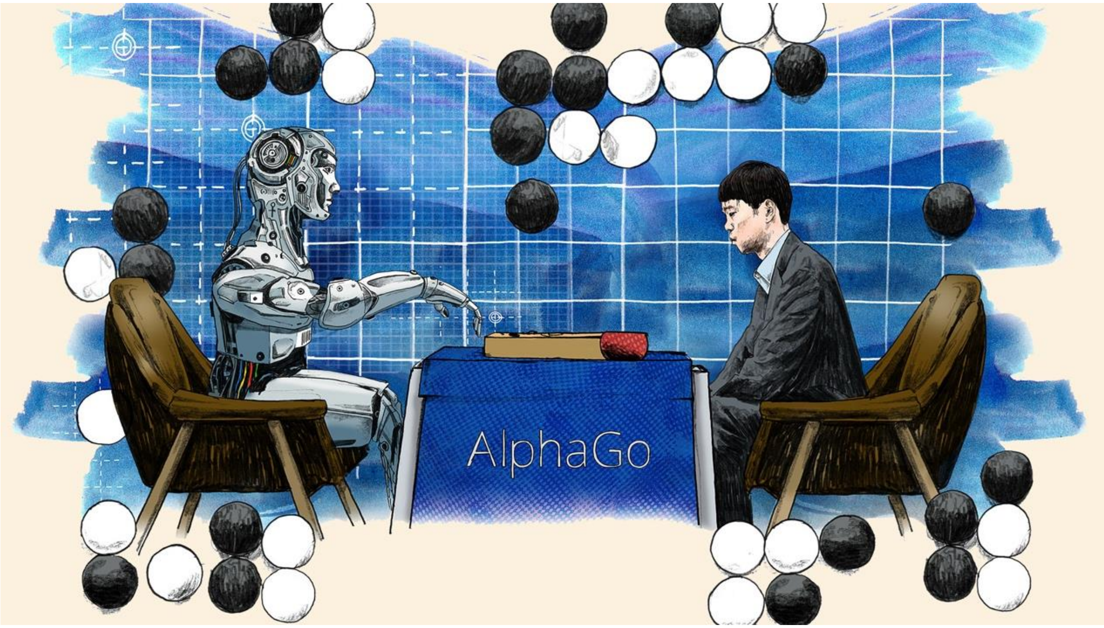
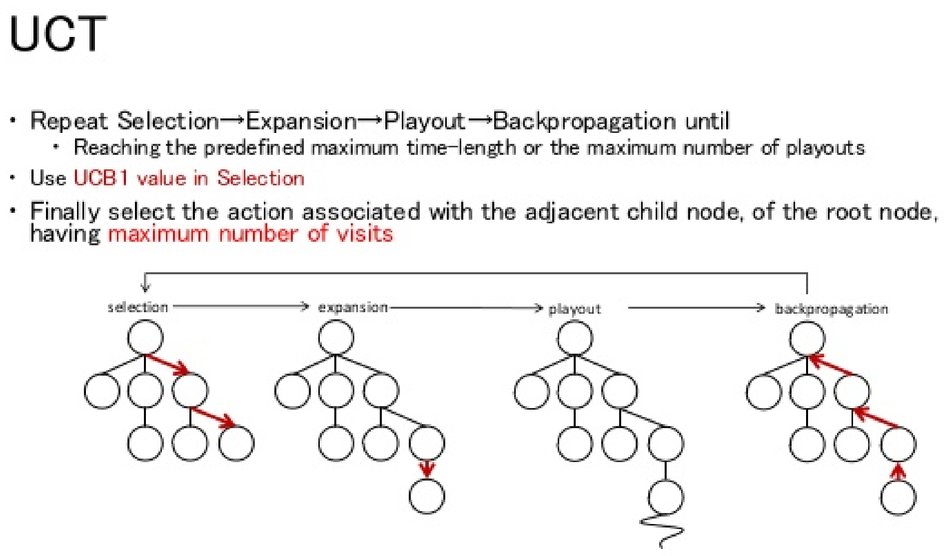


Figure 6: RL ranking system of *TaoBao* search engine

Applications: A component of game-playing

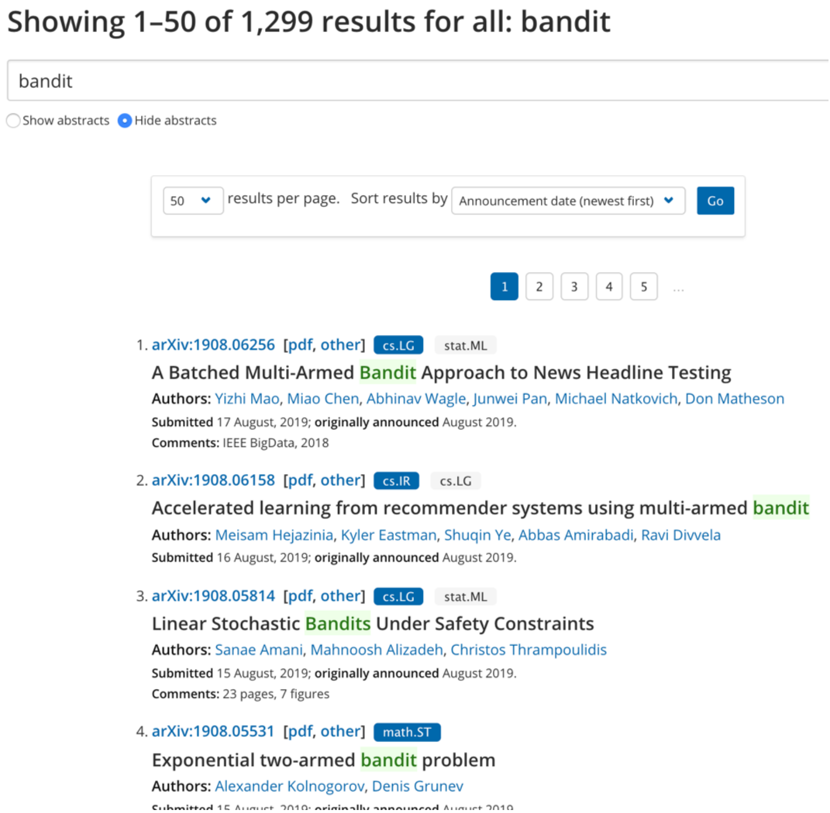
- A component of game-playing algorithms
  - Monte-Carlo tree search (MCTS) – AlphaGo [Silver et al. (2016)]
  - UCT algorithm [Kocsis and Szepesvari (2006)]
  - Drives its search uses a bandit algorithm at each node

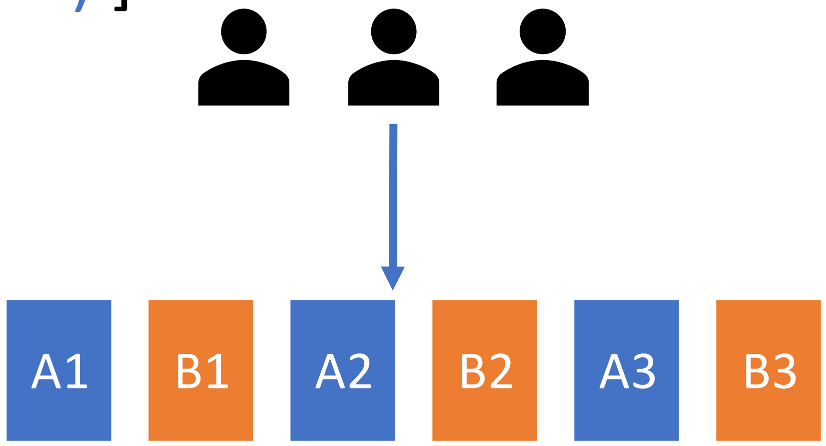




Applications: Select policies

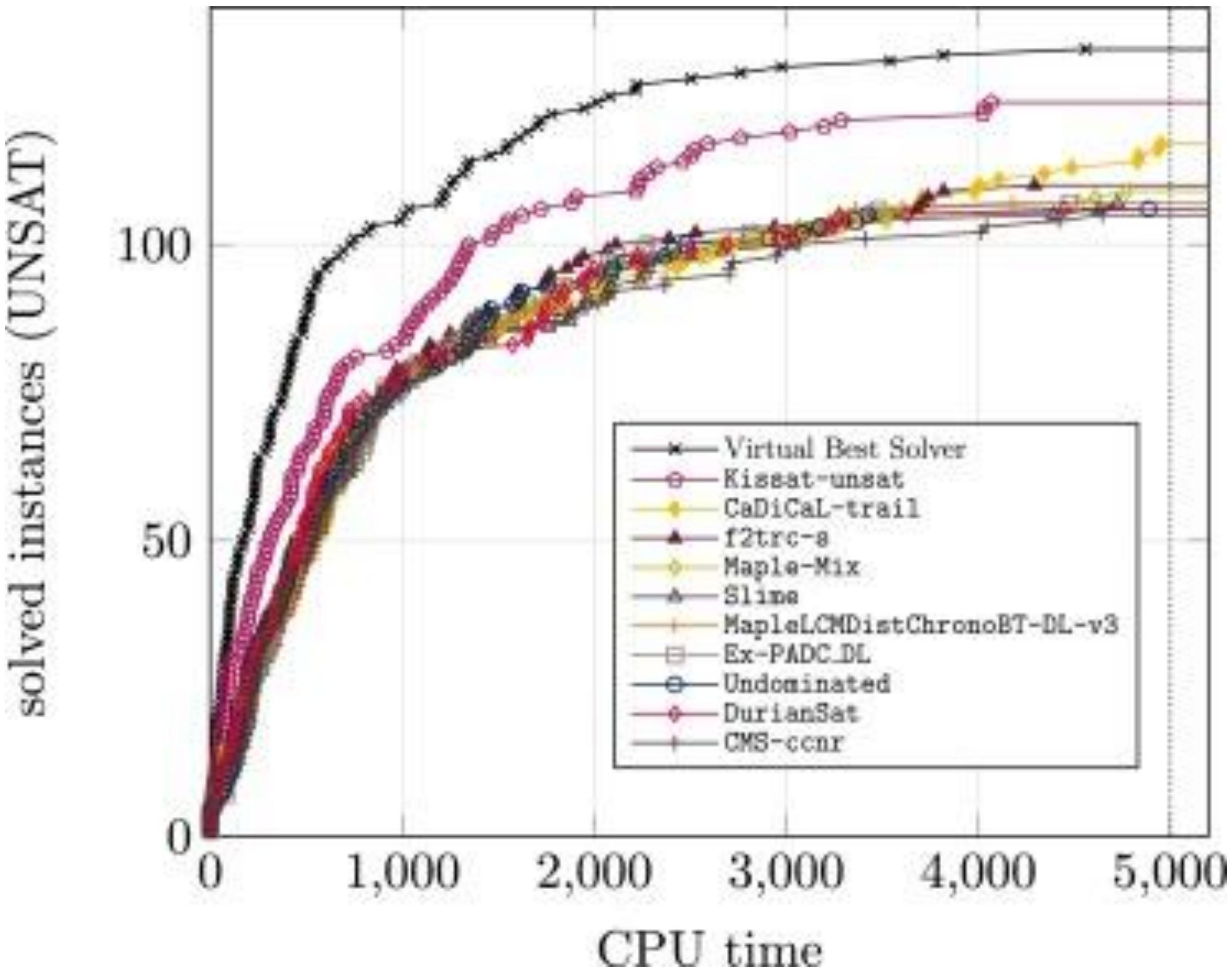
- Select the best ranking policy [Yue et al. (2012)]
  - Online interleaving





Combined Ranker A&B

Applications: SAT solvers



Other applications

- Clinical trials [Villar et al. (2015)]
- Network routing [Le et al. (2014)]
- Experimental design [Rafferty et al. (2018)]
- Hyperparameter tuning [Li et al. (2017)]
- A/B testing [many]
- Ad placement [Yu et al. (2016)]
- Dynamic pricing (eg.,for Amazon products) [Babaioff et al. (2015)]
- Ranking (eg.,for search) [Radlinski et al. (2008)]
- Waiting problems (when to auto-logout your computer) [Lattimore et al. (2014)]
- Resource allocation [Larrnaaga et al. (2016)]

Finite-armed stochastic bandits

Setting: Finite-armed stochastic bandits

items/products/movies/news/…

CTR/profit/…

- There are L arms
  - Each arm a has an unknown reward distribution υ_a with unknown mean $\alpha(a)$
  - The best arm is $a^* = \mathrm{argmax}_a \alpha(a)$

- At each time t
  - The learning agent selects an arm a_t
  - Observes the reward $X_{a_t,t} \sim \upsilon_{a_t}$

bandit feedback

Objective

- Maximize the expected cumulative reward in T rounds

$$\mathbb{E}\left[\sum_{t=1}^{T} \alpha(a_t)\right]$$

- Minimize the regret in T rounds

$$R(T) = T \cdot \alpha(a^*) - \mathbb{E}\left[\sum_{t=1}^{T} \alpha(a_t)\right]$$

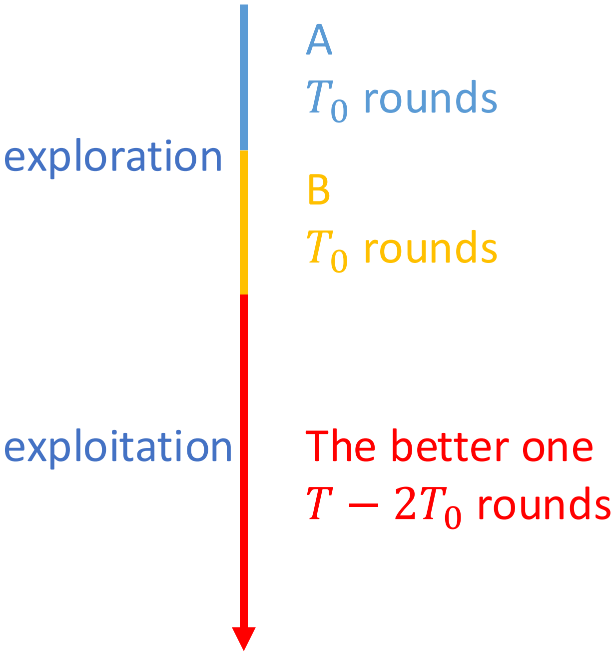
- Balance the trade-off between exploration and exploitation
  - Exploitation: Select arms that yield good results so far
  - Exploration: Select arms that have not been tried much before
- Smaller order of T in $R(T)$ is better

A/B testing

- There are $L = 2$ arms (choices/plans/…)
- Suppose

$$\begin{aligned} \upsilon_A &= \text{Gaussian}(\alpha_A, 1) \\ \upsilon_B &= \text{Gaussian}(\alpha_B, 1) \\ \alpha_A &> \alpha_B, \\ \Delta &= \alpha_A - \alpha_B \end{aligned}$$

- Explore-then-commit algorithm
  - Select each of A and B for T_0 rounds and then select the one with larger sample mean for the remaining $T - 2T_0$ rounds



A/B testing (continued)

- Regret

$$R(T)$$
$$= T_0 \cdot (\alpha_A - \alpha_B) + \mathbb{P}[\hat{\alpha}_A < \hat{\alpha}_B](T - 2T_0)(\alpha_A - \alpha_B)$$
$$< T_0\Delta + T\Delta \cdot \exp\left(-\frac{T_0\Delta^2}{4}\right)$$
$$= O\left(\frac{1}{\Delta}\log T\right)$$

$$T_0 = \left\lceil \frac{4}{\Delta^2} \log\left(\frac{T\Delta^2}{4}\right)\right\rceil$$

need the knowledge of Δ

- $R(T) = \Omega(T\Delta)$ if $T_0 = \frac{1}{5}T$
- $R(T) = \Omega(T\Delta)$ if $T_0 = 1000$

A
T_0 rounds

B
T_0 rounds

The better one
$T - 2T_0$ rounds

A/B testing (continued)

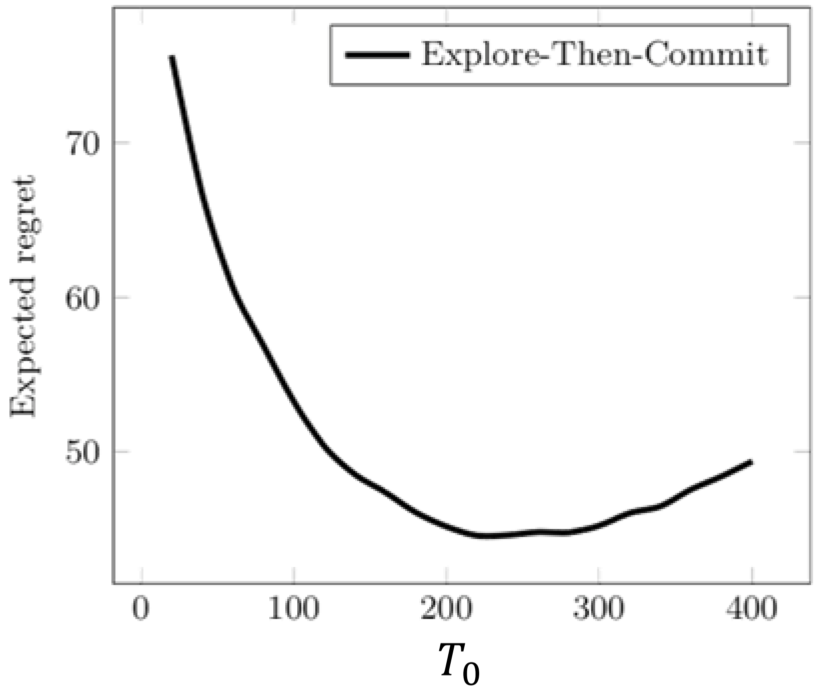


Figure 6.2 Expected regret for Explore-Then-Commit over 10^5 trials on a Gaussian bandit with means $\mu_1 = 0, \mu_2 = -1/10$

- Lattimore and Szepesvári (2018)

Epsilon-greedy algorithm

- For each time t
  - $\epsilon_t \in (0,1)$
  - With probability ϵ_t, randomly choose an arm
  - With probability $1 - \epsilon_t$, choose the one with highest sample mean

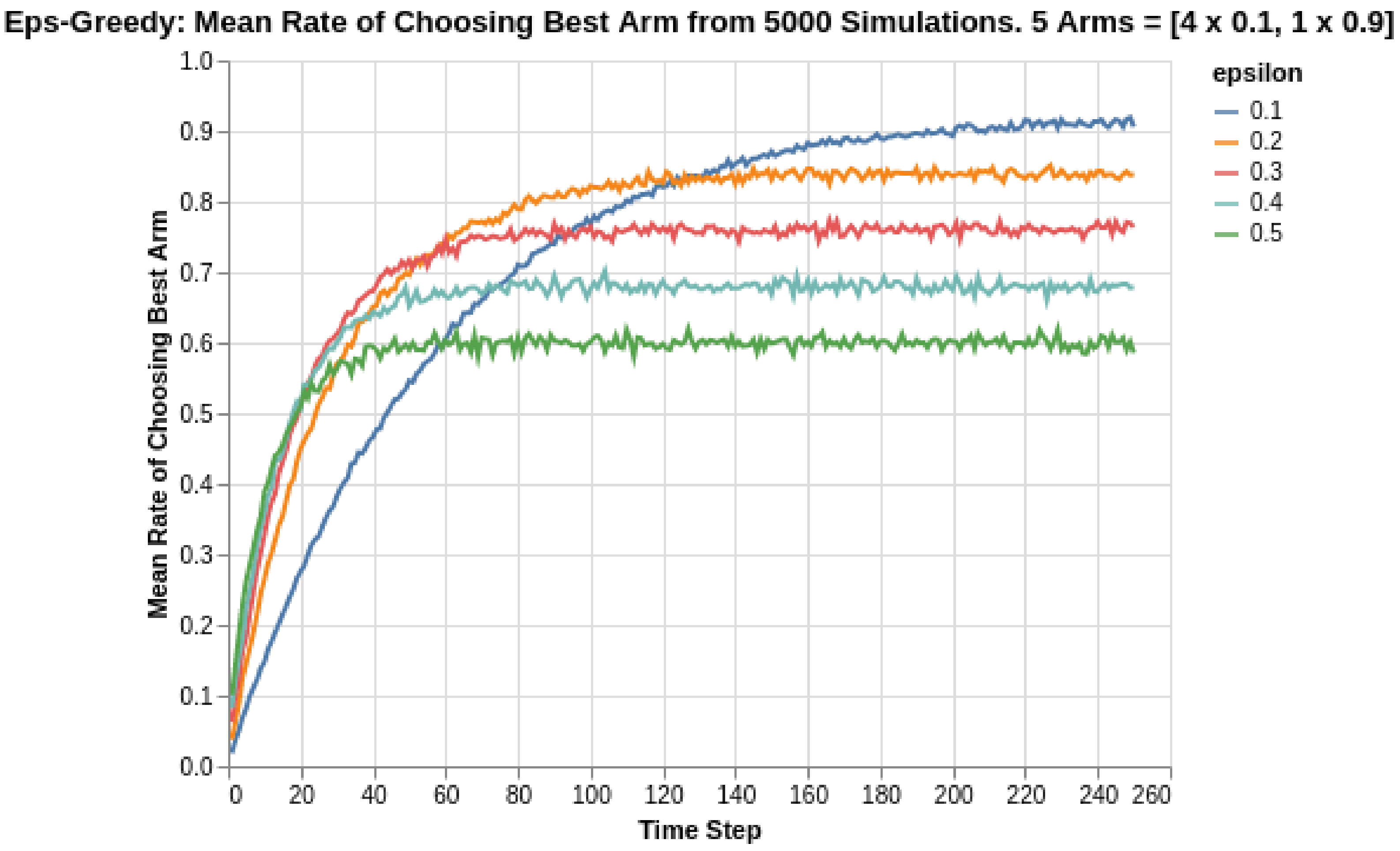
exploration

exploitation

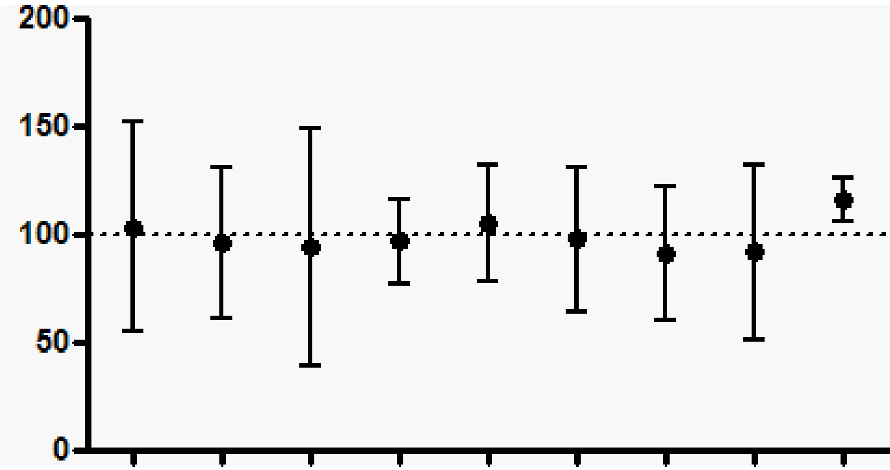
- When $\epsilon_t = \min\left\{1, \frac{c}{t\Delta^2}\right\}$, regret $R(T) = O\left(\frac{L}{\Delta}\log T\right)$

need the knowledge of Δ

Epsilon-greedy algorithm 2



UCB – Upper confidence bound [Auer et al.(2002)]



- With high probability

sample mean

round t

$$\alpha_a \epsilon \left[\hat{\alpha}_a(t) - \sqrt{\frac{2 \log t}{T_a(t)}}, \hat{\alpha}_a(t) + \sqrt{\frac{2 \log t}{T_a(t)}} \right]$$

Hoeffding's inequality

selection times of arm a
till round t

- Principle: optimism in face of uncertainty
- UCB policy:

exploration

$$a_t = \text{argmax}_a \, \hat{\alpha}_a + \sqrt{\frac{2 \log t}{T_a(t)}}$$

exploitation

UCB – Upper confidence bound 2

- Regret

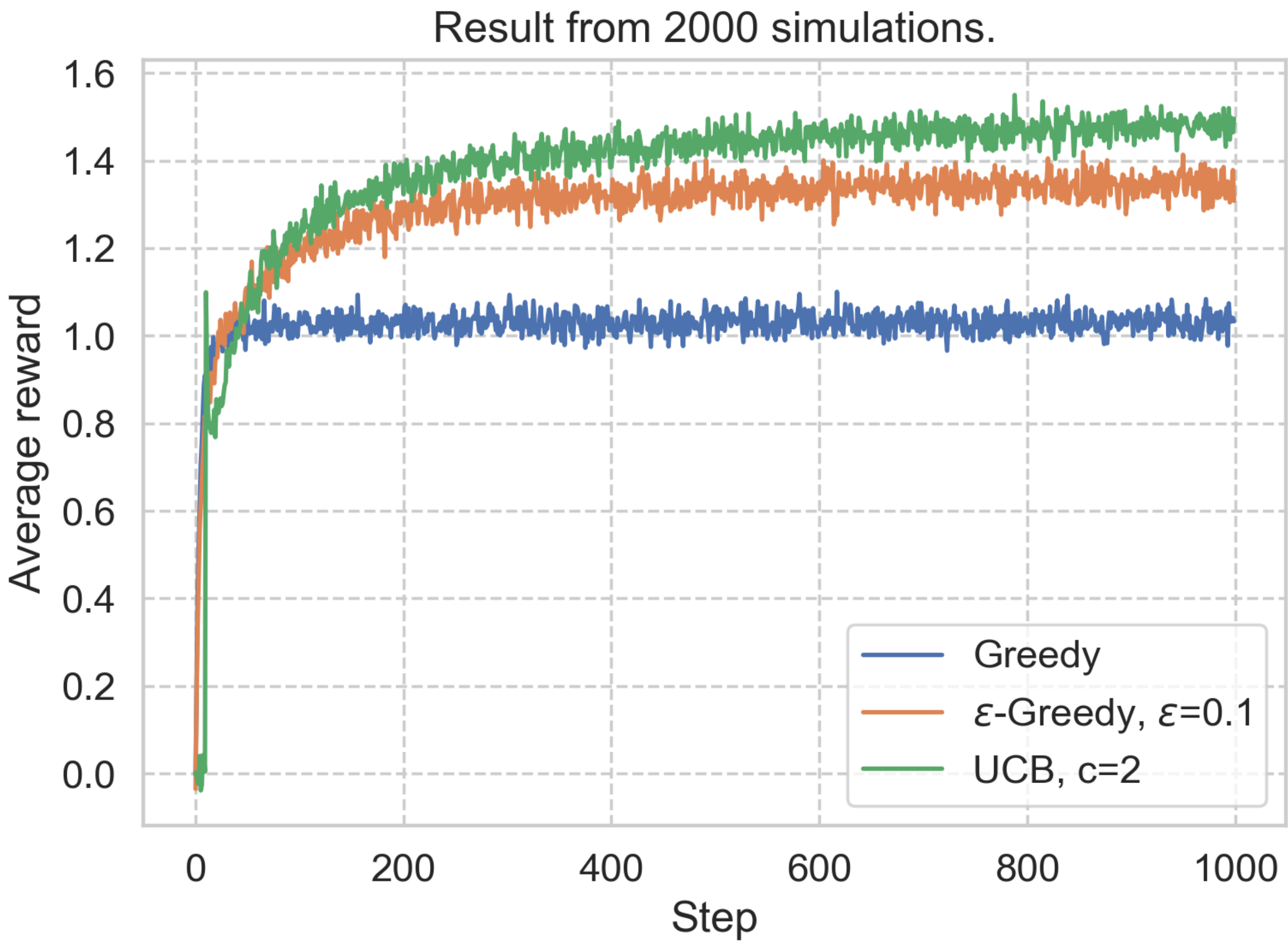
$$R(T) = O\left(\frac{L}{\Delta} \log T\right)$$

- Proof sketch
  - Under good event (w/ high probability)
  - If arm a is pulled, then

$$\alpha(a^*) \leq \mathrm{UCB}_{a^*} \leq \mathrm{UCB}_a \leq \alpha(a) + 2\ \mathrm{radius}_a$$

  - $\Longrightarrow \sqrt{\frac{2 \log t}{T_a(t)}} = \mathrm{radius}_a \geq \frac{\alpha(a^*) - \alpha(a)}{2}$
  - $\Longrightarrow T_a(t) \leq \frac{8 \log t}{\Delta_a^2}$

UCB – Upper confidence bound 3



Thompson sampling [Agrawal and Goyal (2013)]

- Assume each arm has prior Gaussian$(0,1)$
- Then posterior distribution for α_a is

$$\text{Gaussian}\left(\hat{\alpha}_a(t), \frac{1}{1+T_a(t)}\right)$$

- Sample

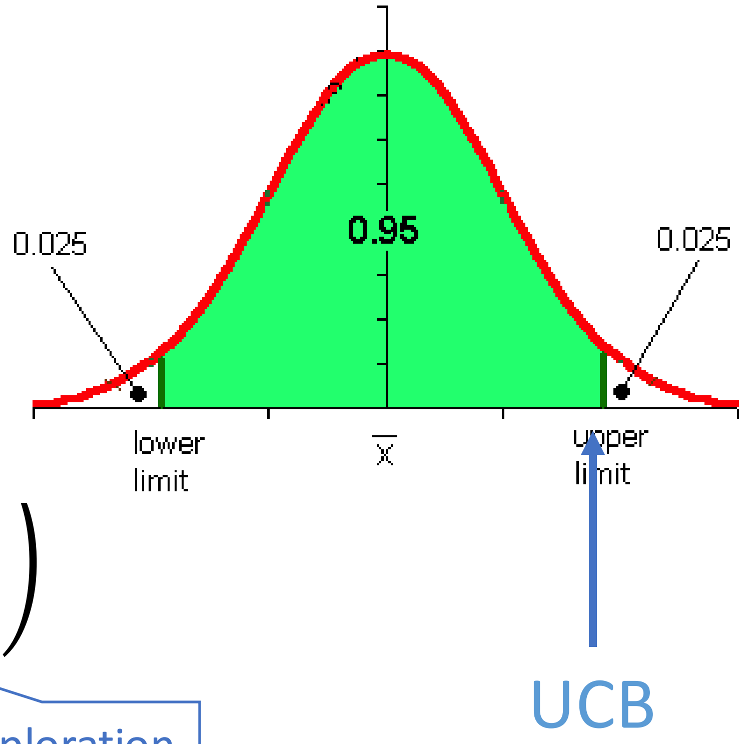
exploitation

$$\tilde{\alpha}_a(t) \sim \text{Gaussian}\left(\hat{\alpha}_a(t), \frac{1}{1+T_a(t)}\right)$$

exploration

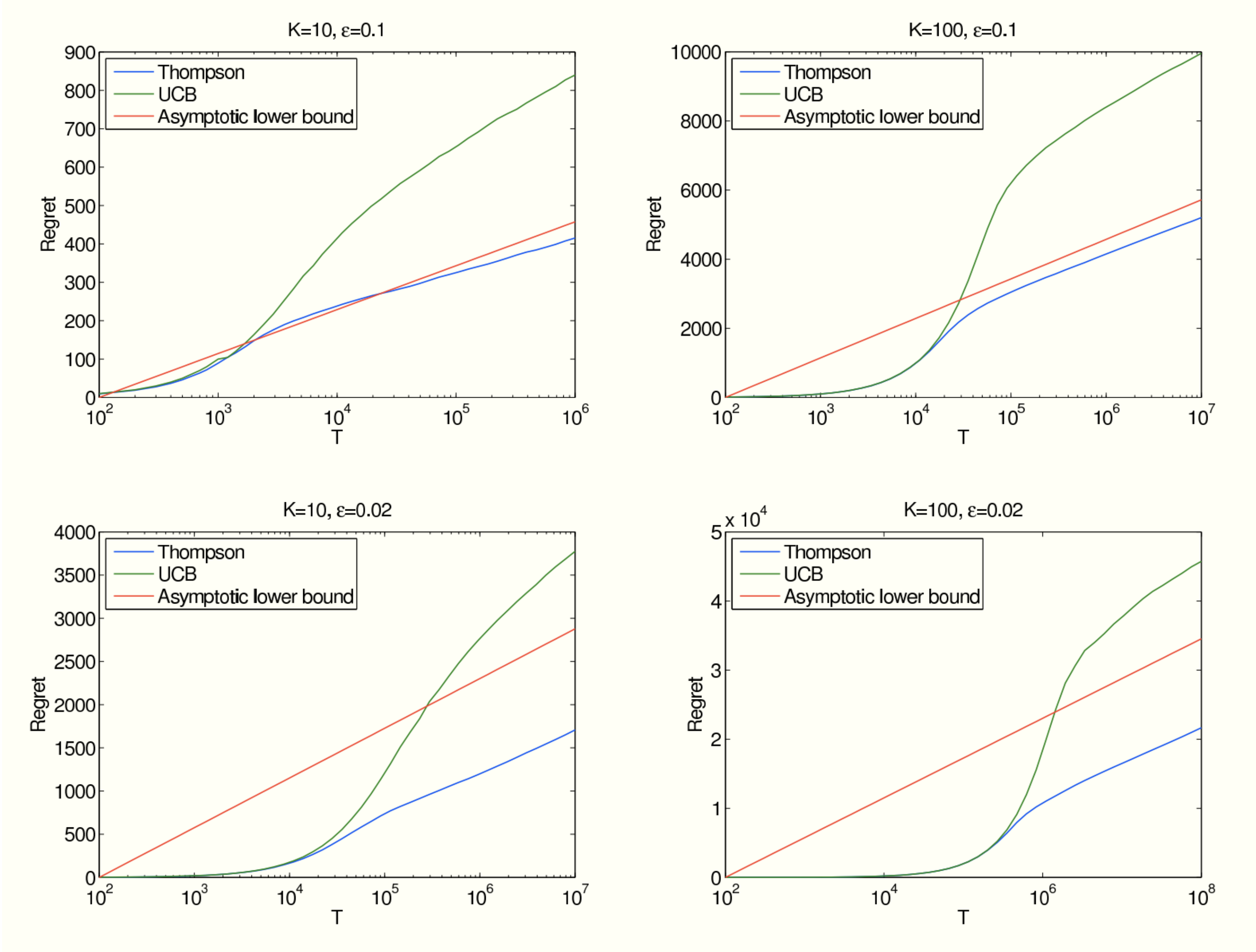
and select

$$a_t = \text{argmax}_a\, \tilde{\alpha}_a(t)$$

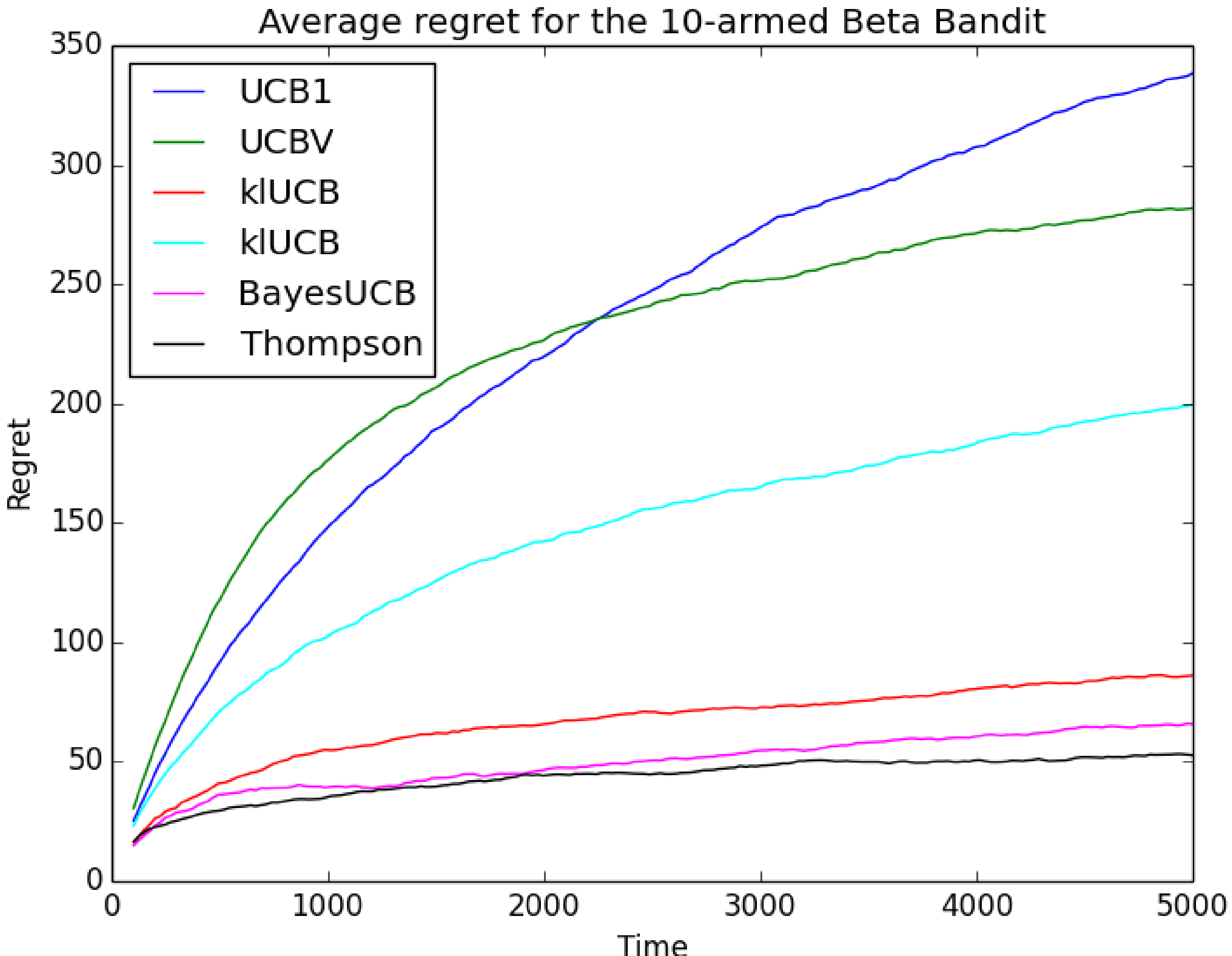


Thompson sampling 2

- Also has optimal regret bound
- Outperform UCB [Chapelle and Li (2011)]



Thompson sampling 3



Finite-armed adversarial bandits

Setting: Adversarial MAB

- There are L arms
  - An adversary secretly preselects all loss vectors $\{l_{t,a}\}_{t,a}$ from $[0,1]$
  - The best arm is $a^* = \text{argmin} \sum_{t=1}^{T} l_{t,a}$

Setting: Adversarial MAB 2

- At each time t
  - The learning agent selects one arm a_t
  - Observe the loss l_{t,a_t}
- Objective:
  - Minimize the expected cumulative loss in T rounds $\mathbb{E}\left[\sum_{t=1}^{T} l_{t,a_t}\right]$
  - Minimize the regret in T rounds

$$R(T) = \mathbb{E}\left[\sum_{t=1}^{T} l_{t,a_t}\right] - \min_a \sum_{t=1}^{T} l_{t,a}$$

- Balance the trade-off between exploration and exploitation
  - Exploitation: Select arms that yield good results so far
  - Exploration: Select arms that have not been tried much before

Exp3: Exponential Weight Algorithm for Exploration and Exploitation

- Importance-weight estimator

$$\hat{l}_{t,a} = \frac{\mathbb{I}\{a_t = a\} \cdot l_{t,a_t}}{\mathbb{P}(a_t = a)}$$

- For each time t
  - Calculate the sampling distribution

Learning rate

$$\mathbb{P}(a_t = a) = \frac{\exp(-\eta \hat{L}_{t-1,a})}{\sum_{b=1}^{n} \exp(-\eta \hat{L}_{t-1,b})}$$

Exponential weighting

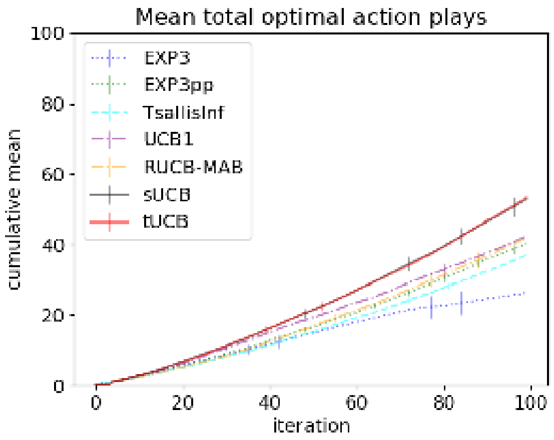
  - Sample $a_t \sim \mathbb{P}(a_t = a)$ and observe l_{t,a_t}
  - Calculate $\hat{L}_{t,a} = \sum_{s=1}^{t} \hat{l}_{t,a}$
- Regret bound $O(\sqrt{LT \log L})$

Comparison between Stochastic and Adversarial Environments

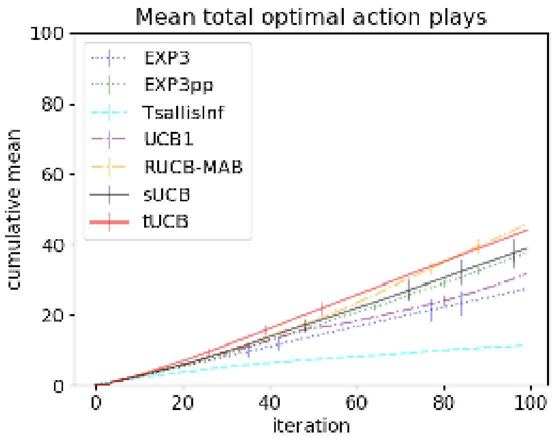
- Stochastic
- Reward fixed distribution on $[0,1]$ with fixed mean
- Best arm $a^* = \operatorname{argmax} \alpha(a)$
- Regret bound $O(\log T)$
- Runs in adversarial setting
  - Regret may not even converge

- Adversarial
- Loss arbitrary on $[0,1]$
- Best arm $a^* = \operatorname{argmin} \sum_{t=1}^{T} l_{t,a}$
- Regret bound $O(\sqrt{T})$
- Runs in stochastic setting
  - Regret bound $O(\sqrt{T})$

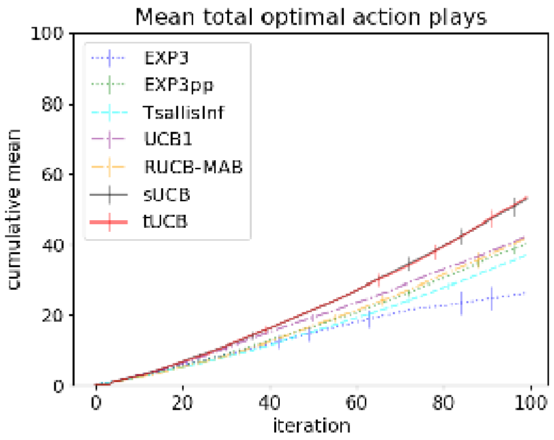
Performance comparisons



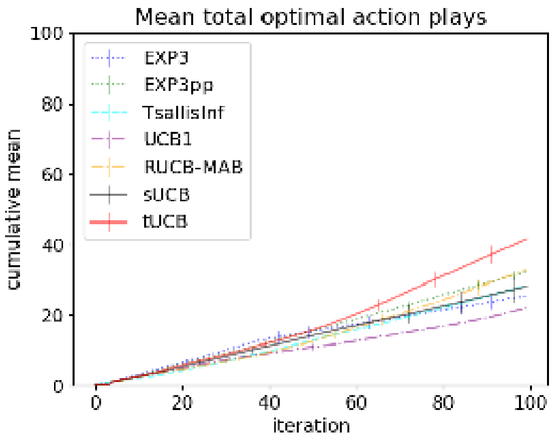
(a) No adversary



(b) $\varepsilon = 0.05$



(a) No adversary



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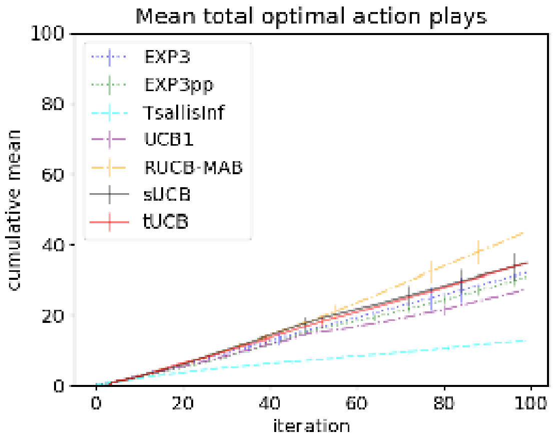
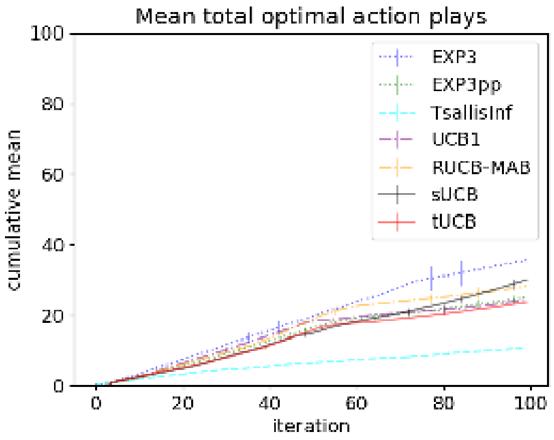
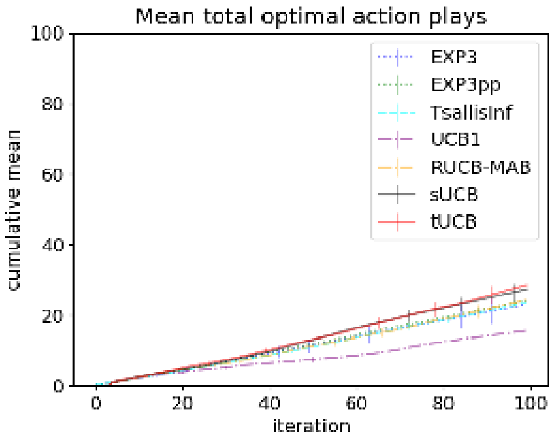


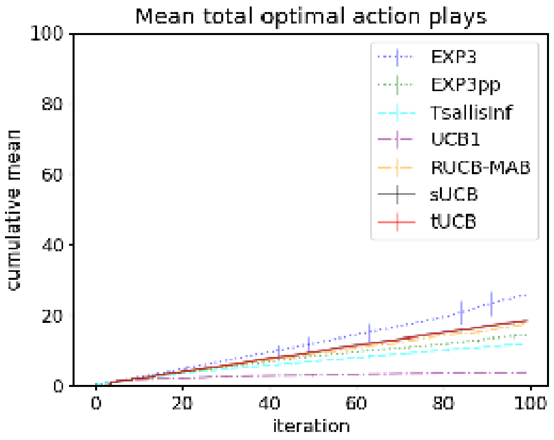
(c) $\varepsilon = 0.1$



(d) $\varepsilon = 0.3$



(c) $\varepsilon = 0.1$



(d) $\varepsilon = 0.3$

Linear bandits

Setting

- At each time t
  - The learning agent receives $D_t \subset \mathbb{R}^d$
  - Selects an arm $a_t \in D_t \subset \mathbb{R}^d$
  - Receives a random reward $X_{a_t,t} \sim \upsilon_{a_t}$ with mean

$$\alpha(a_t) = \theta^\top a_t$$

  for some fixed but unknown weight vector θ

- Allow for large-scale applications

a set of items/videos/news/…

bandit feedback w/ linear structure

LinUCB [Li et al. (2010)]

- The observed feedback is

$$\{(a_1, X_{a_1,1}), (a_2, X_{a_2,2}), \dots, (a_t, X_{a_t,t}), \dots\}$$

- Let

$$V_t = I + \sum_{s=1}^{t} a_s a_s^T, b_t = \sum_{s=1}^{t} X_{a_s,s}\, a_s$$

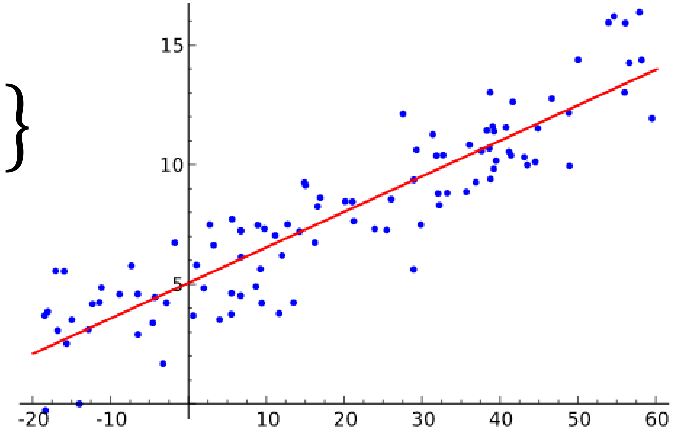
linear regression estimator

exploitation

$$\hat{\theta}_t = V_t^{-1} b_t$$

- With high probability

$$\left\|\theta - \hat{\theta}_t\right\|_{V_t} \leq C\sqrt{\log t}$$



How to understand $\left\|\theta - \hat{\theta}_t\right\|_{V_t} \le C\sqrt{\log t}$

- $\|x\|_{V_t} = \sqrt{x^\top V_t x}$
- If $V_t = \begin{bmatrix} T_1 & \\ & T_2 \end{bmatrix}$, then $\|x\|_{V_t} = \sqrt{T_1|x_1|^2 + T_2|x_2|^2}$
  - When $a_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, a_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

$$V_t = I + \sum_{s=1}^{t} a_s a_s^T = \begin{bmatrix} 1 + T_{a_1}(t) & \\ & 1 + T_{a_2}(t) \end{bmatrix}$$

implies

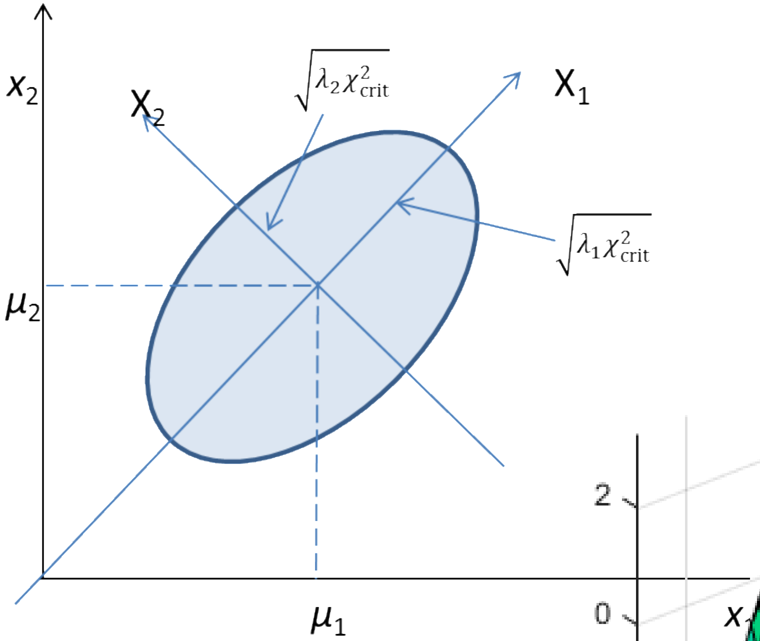
$$|\alpha(a_1) - \hat{\alpha}(a_1)| = \left|\theta_1 - \hat{\theta}_1\right| \le C\sqrt{\frac{\log t}{1 + T_{a_1}(t)}}$$

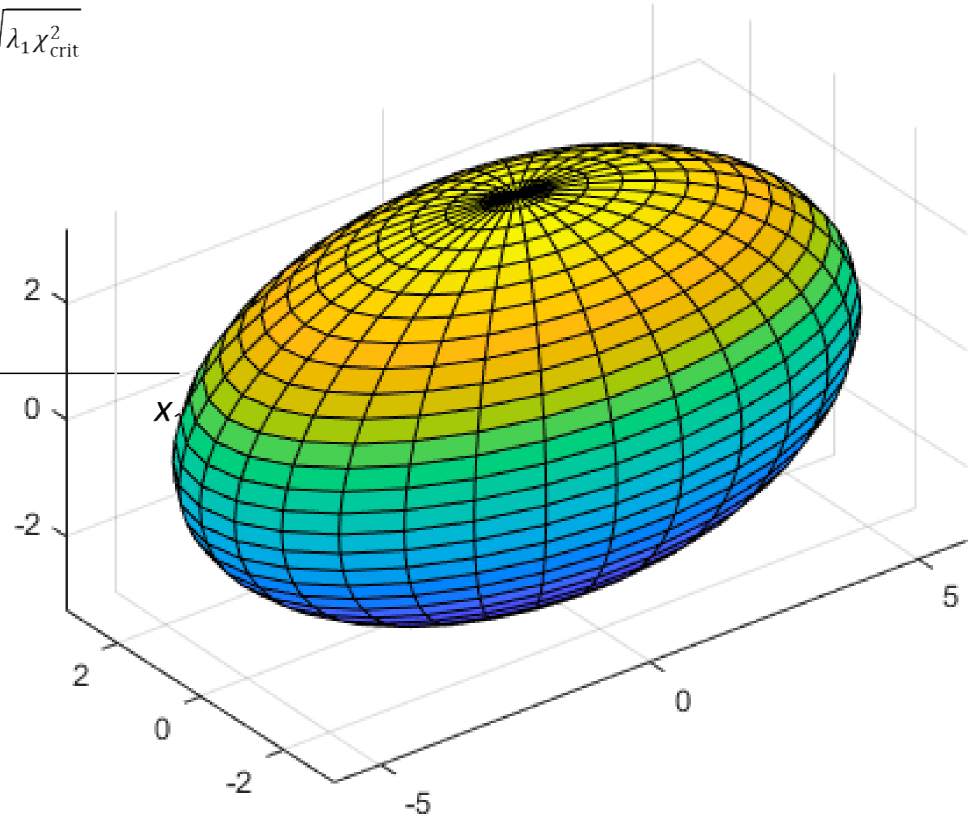
$$|\alpha(a_2) - \hat{\alpha}(a_2)| = \left|\theta_2 - \hat{\theta}_2\right| \le C\sqrt{\frac{\log t}{1 + T_{a_2}(t)}}$$

UCB!

How to understand $\|\theta - \hat{\theta}_t\|_{V_t} \leq C\sqrt{\log t}$

- For general V_t
- Confidence ellipse





- Confidence ellipsoid
  - narrower direction means less uncertainty

How to use it

- With high probability

$$\left|\alpha(a) - \hat{\theta}_t^{\top} a\right| \le C\sqrt{\log t}\,\|a\|_{V_t^{-1}}$$

- Select

$$a_{t+1} = \mathrm{argmax}_a\, \underbrace{\hat{\theta}_t^{\top} a}_{\text{exploitation}} + \underbrace{C\sqrt{\log t}\,\|a\|_{V_t^{-1}}}_{\text{exploration}}$$

- Regret

$$R(T) = O\left(d\sqrt{T}\log T\right)$$

LinTS [Agrawal and Goyal (2013b)]

- Suppose θ has prior $\mathrm{Gaussian}(0, I)$
- Then the posterior distribution for θ is

$$\mathrm{Gaussian}\left(\hat{\theta}, V_t^{-1}\right)$$

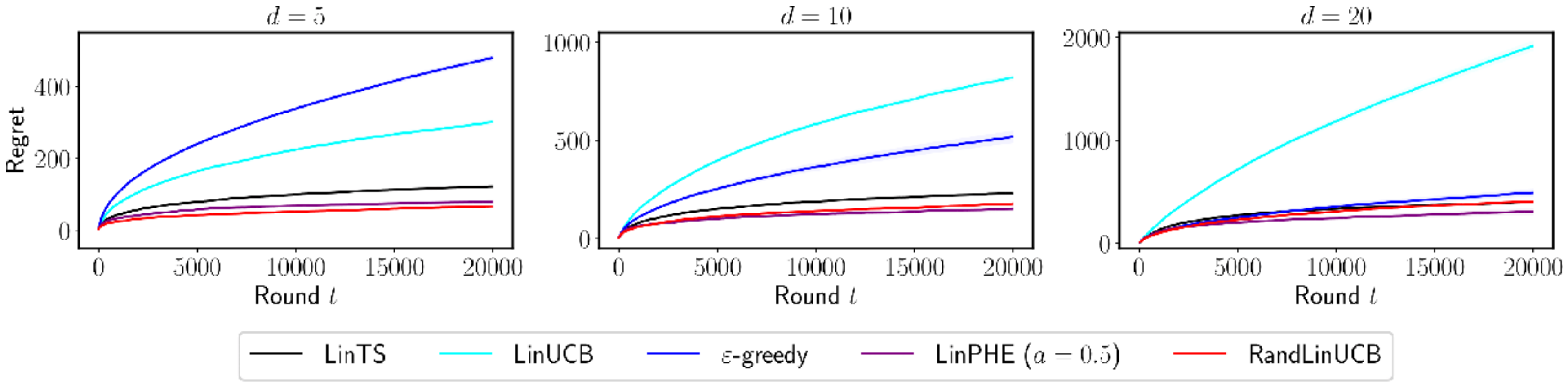
- Draw a random sample

$$\tilde{\theta} \sim \mathrm{Gaussian}\left(\hat{\theta}, V_t^{-1}\right)$$

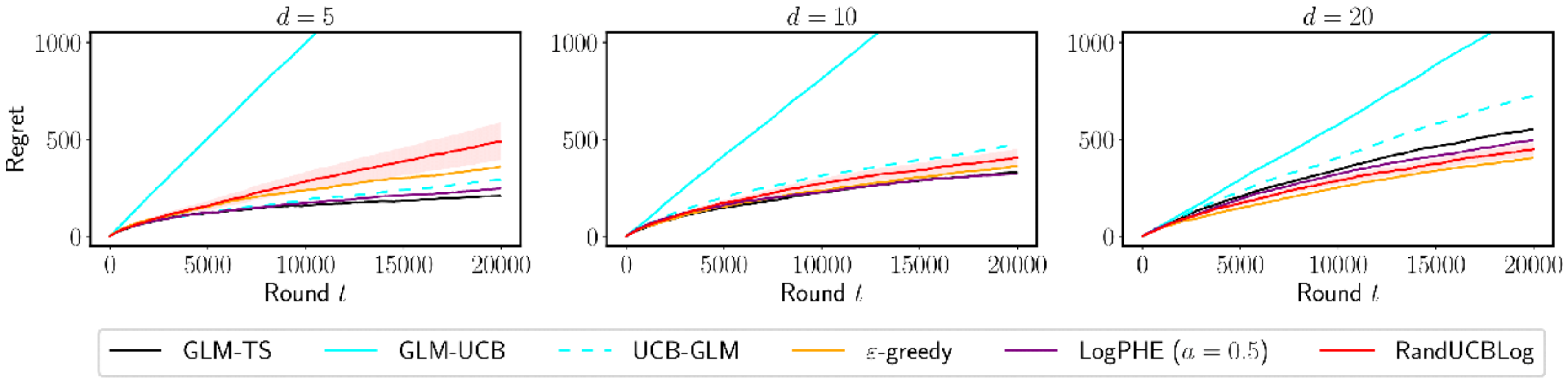
  and select

$$a_{t+1} = \mathrm{argmax}_a\, \tilde{\theta}^\top a$$

Comparisons



(b) Linear bandits of different dimension (d).



(c) Generalized linear bandits of different dimension (d).

Summary

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- What are bandits, and why should you care
  - Many applications
- Finite-armed bandits
  - Explore-then-commit
  - epsilon-greedy
  - UCB: $a_t = \text{argmax}_a \hat{\alpha}_a + \sqrt{\frac{2 \log t}{T_a(t)}}$
  - Thompson sampling: $\tilde{\alpha}_a(t) \sim \text{Gaussian}\left(\hat{\alpha}_a(t), \frac{1}{1+T_a(t)}\right)$
  - EXP3
- Linear bandits
  - LinUCB, LinTS

Questions?

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