Lecture 2: Search

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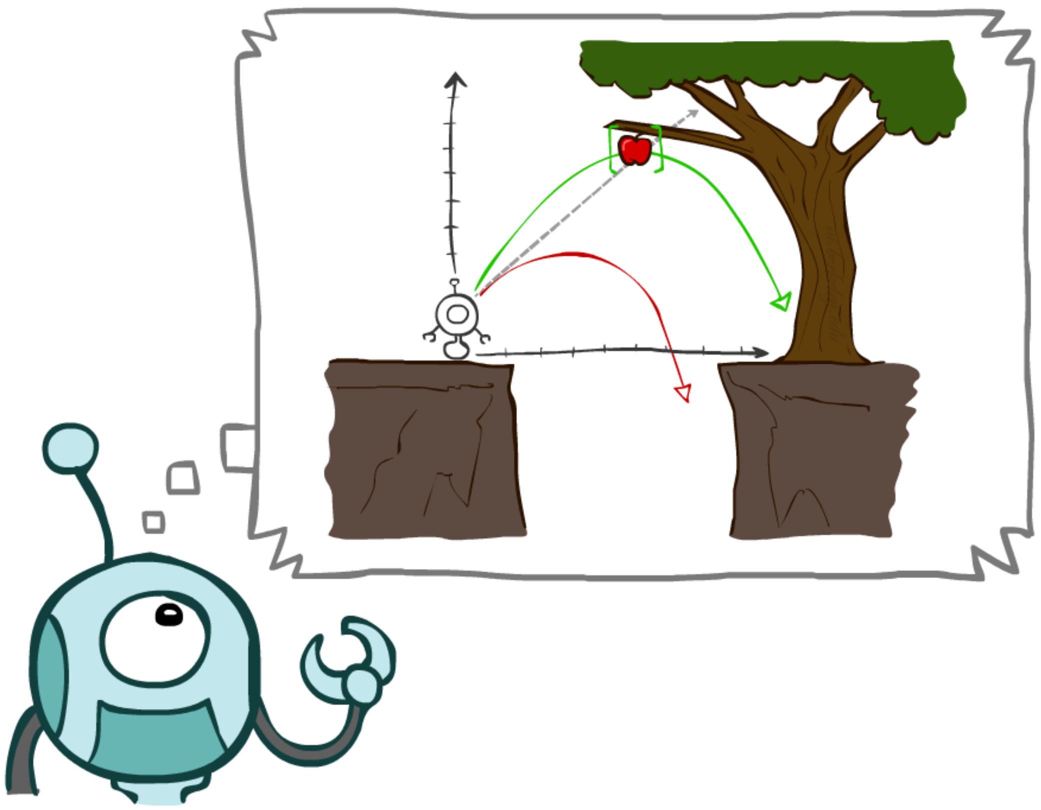
https://shuaili8.github.io/Teaching/CS3317/index.html

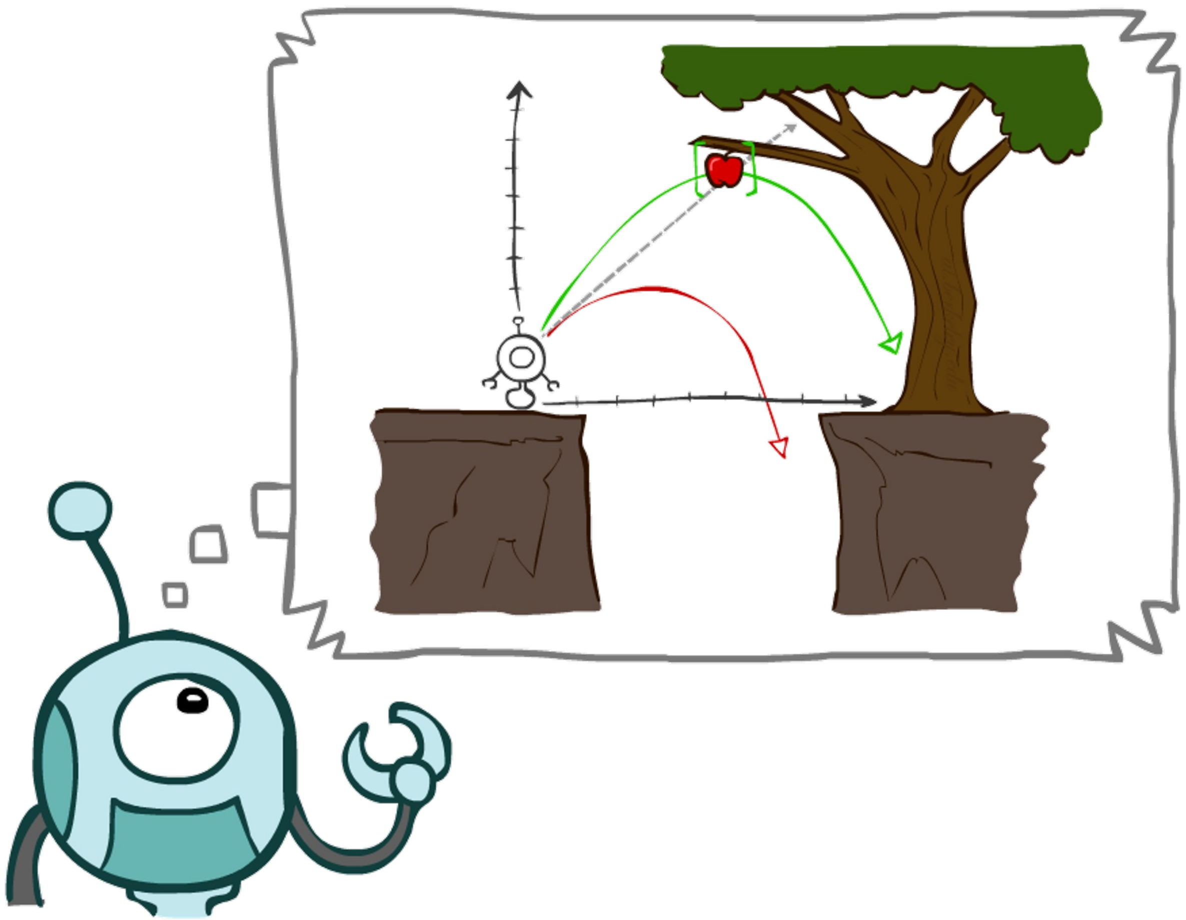
Part of slide credits: CMU AI & http://ai.berkeley.edu

Today

- Agents that Plan Ahead
- Search Problems
- Uninformed Search Methods
  - Depth-First Search
  - Breadth-First Search
  - Uniform-Cost Search
- Informed Search
  - Heuristics
  - Greedy Search
  - A* Search
  - Graph Search

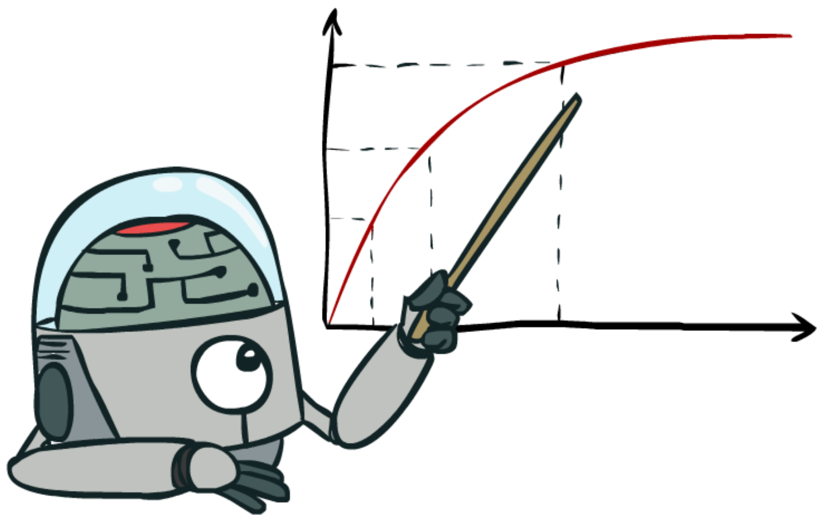


Agents that Plan



Rationality

- What is rational depends on:
  - Performance measure
  - Agent's prior knowledge of environment
  - Actions available to agent
  - Percept/sensor sequence to date

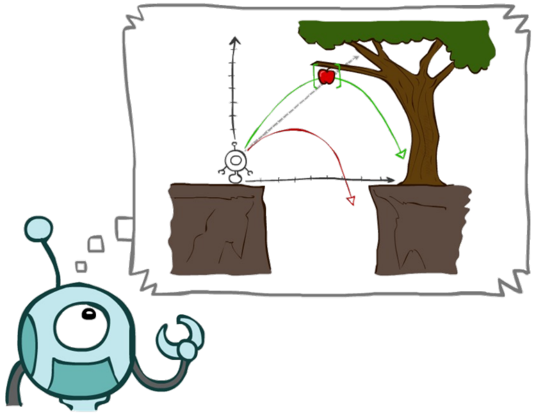


- Being rational means **maximizing your expected utility**

Rational Agents

- Are rational agents ***omniscient***? 无所不知的
  - No – they are limited by the available percepts
- Are rational agents ***clairvoyant***? 透视的
  - No – they may lack knowledge of the environment dynamics
- Do rational agents ***explore*** and ***learn***?
  - Yes – in unknown environments these are essential
- So rational agents are not necessarily successful, but they are ***autonomous*** (i.e., control their own behavior)

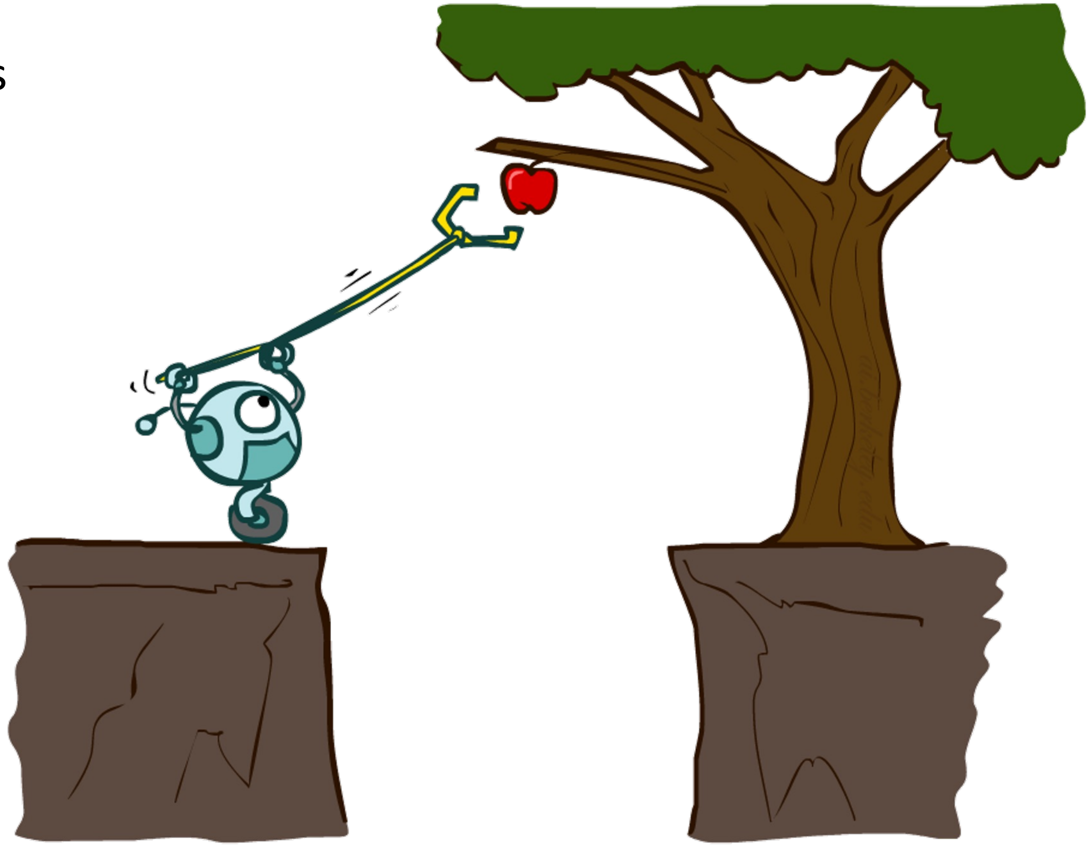
Planning Agents



- Planning agents:
  - Ask “what if”
  - Decisions based on (hypothesized or predicted) consequences of actions
  - Must have a transition model of how the world evolves in response to actions
  - Must formulate a goal (test)
  - Consider how the world WOULD BE
- Spectrum of deliberativeness:
  - Generate complete, optimal plan offline, then execute
  - Generate a simple, greedy plan, start executing, replan when something goes wrong

- Optimal vs. complete planning
- Planning vs. replanning

[Demo: re-planning (L2D3)]
[Demo: mastermind (L2D4)]



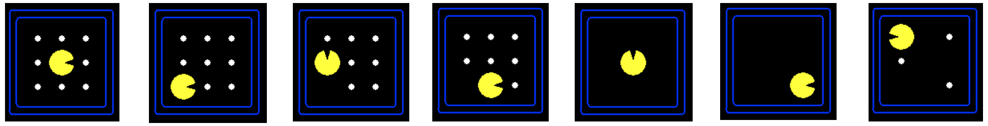
Video of Demo Replanning

Video of Demo Mastermind

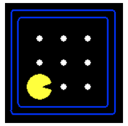
Search Problems

Search Problems

- A search problem consists of:
  - A state space

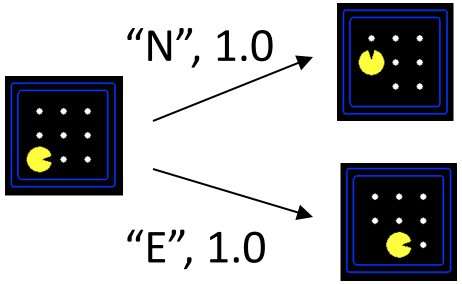


  - For each state, a set Actions(s) of successors/actions



{N, E}

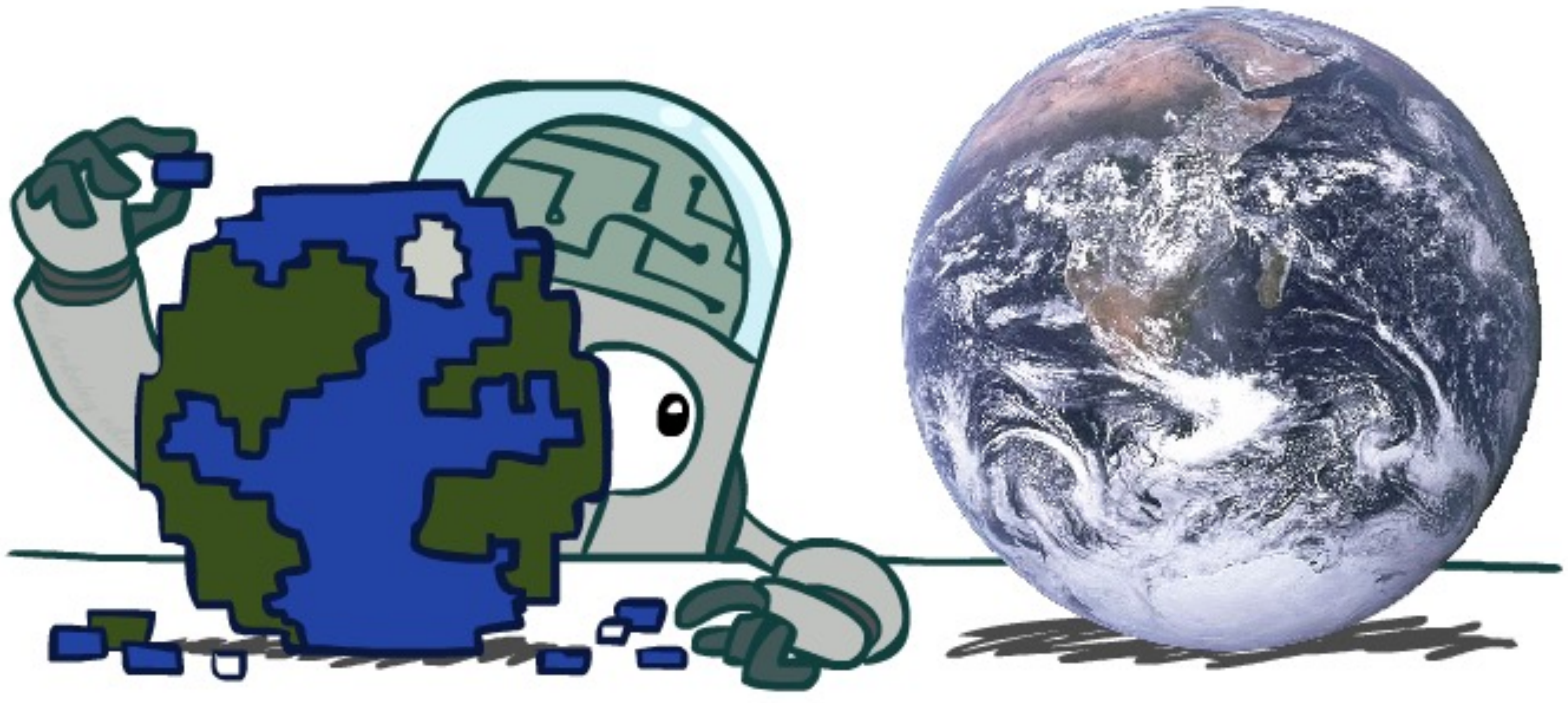
  - A successor function
    - A transition model T(s,a)
    - A step cost(reward) function c(s,a,s')



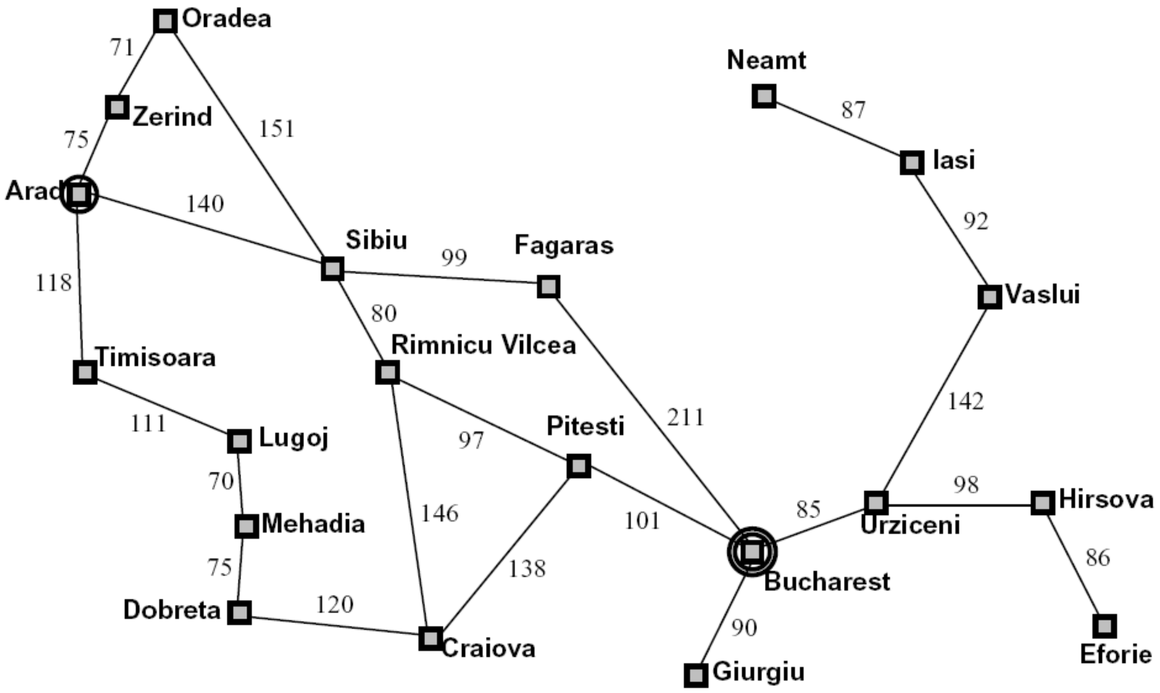
  - A start state and a goal test

- A solution is a sequence of actions (a plan) which transforms the start state to a goal state

Search Problems Are Models



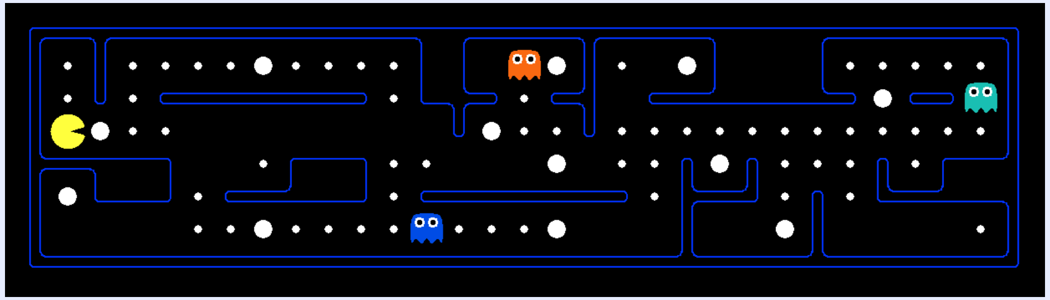
Example: Traveling in Romania



- State space:
  - Cities
- Successor function:
  - Roads: Go to adjacent city with cost = distance
- Start state:
  - Arad
- Goal test:
  - Is state == Bucharest?
- Solution?

What’s in a State Space?

The world state includes every last detail of the environment



A search state keeps only the details needed for planning (abstraction)

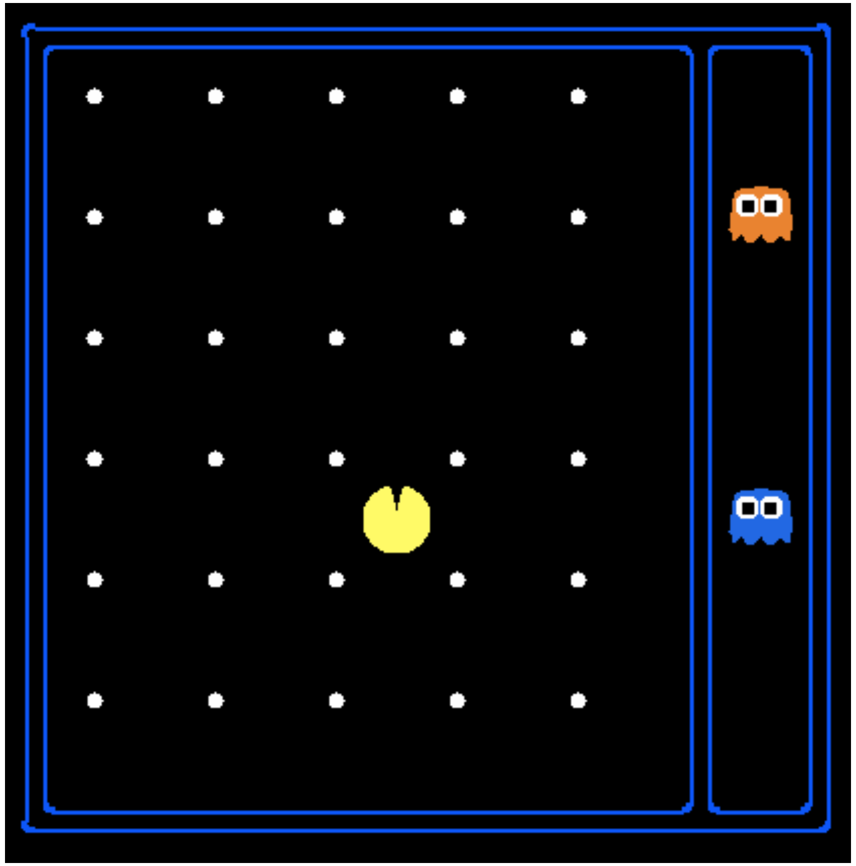
- Problem: Pathing
  - States: (x,y) location
  - Actions: NSEW
  - Successor: update location only
  - Goal test: is (x,y)=END

- Problem: Eat-All-Dots
  - States: {(x,y), dot booleans}
  - Actions: NSEW
  - Successor: update location and possibly a dot boolean
  - Goal test: dots all false

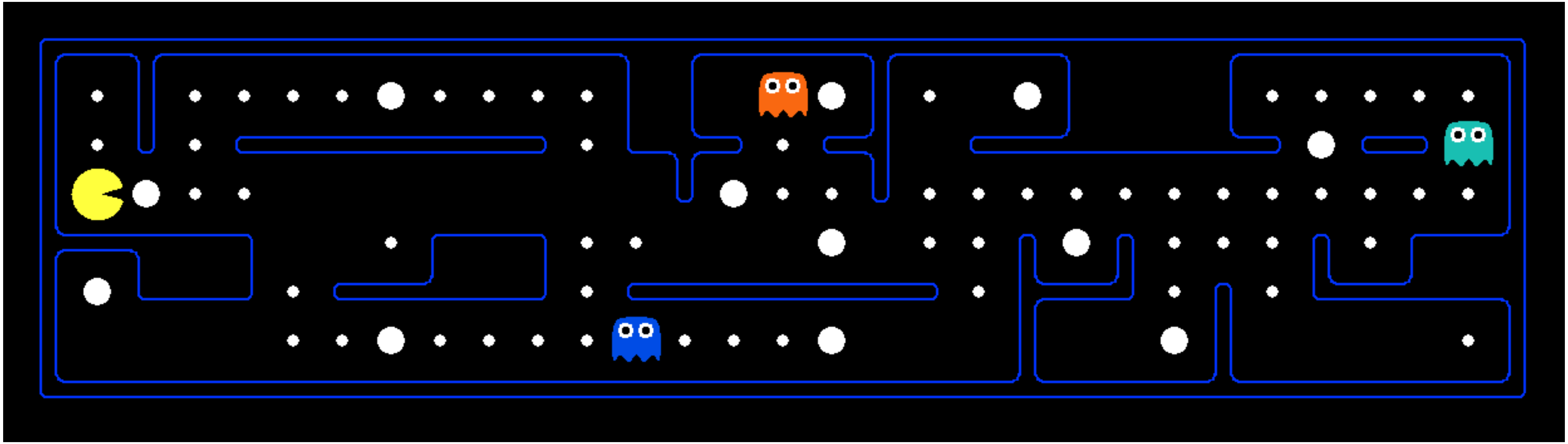
State Space Sizes?

- World state:
  - Agent positions: 120
  - Food count: 30
  - Ghost positions: 12
  - Agent facing: NSEW

- How many
  - World states?
    $120 \times (2^{30}) \times (12^{2}) \times 4$
  - States for pathing?
    120
  - States for eat-all-dots?
    $120 \times (2^{30})$

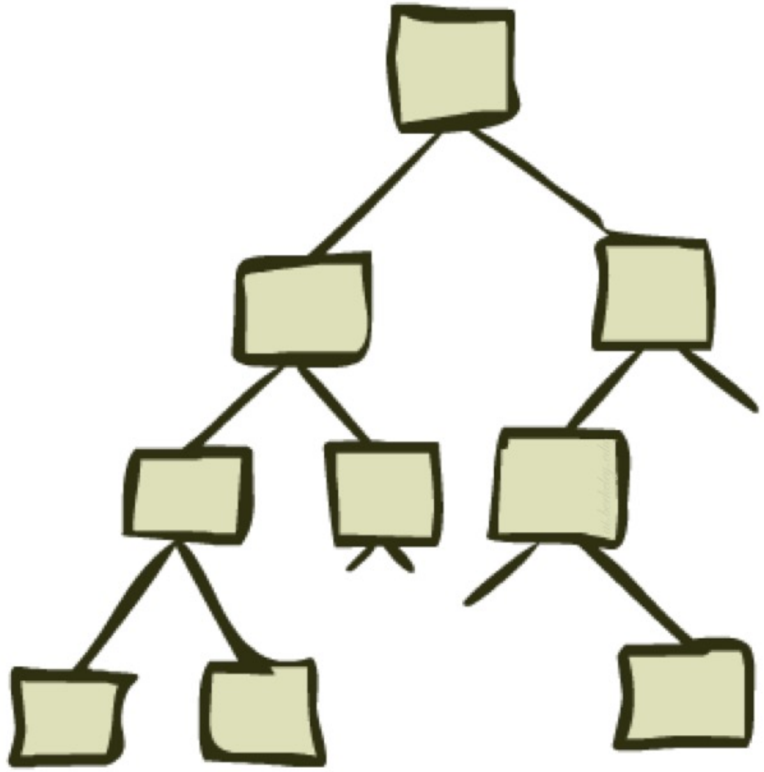


Safe Passage



- Problem: eat all dots while keeping the ghosts perma-scared
- What does the state space have to specify?
  - (agent position, dot booleans, power pellet booleans, remaining scared time)

State Space Graphs and Search Trees

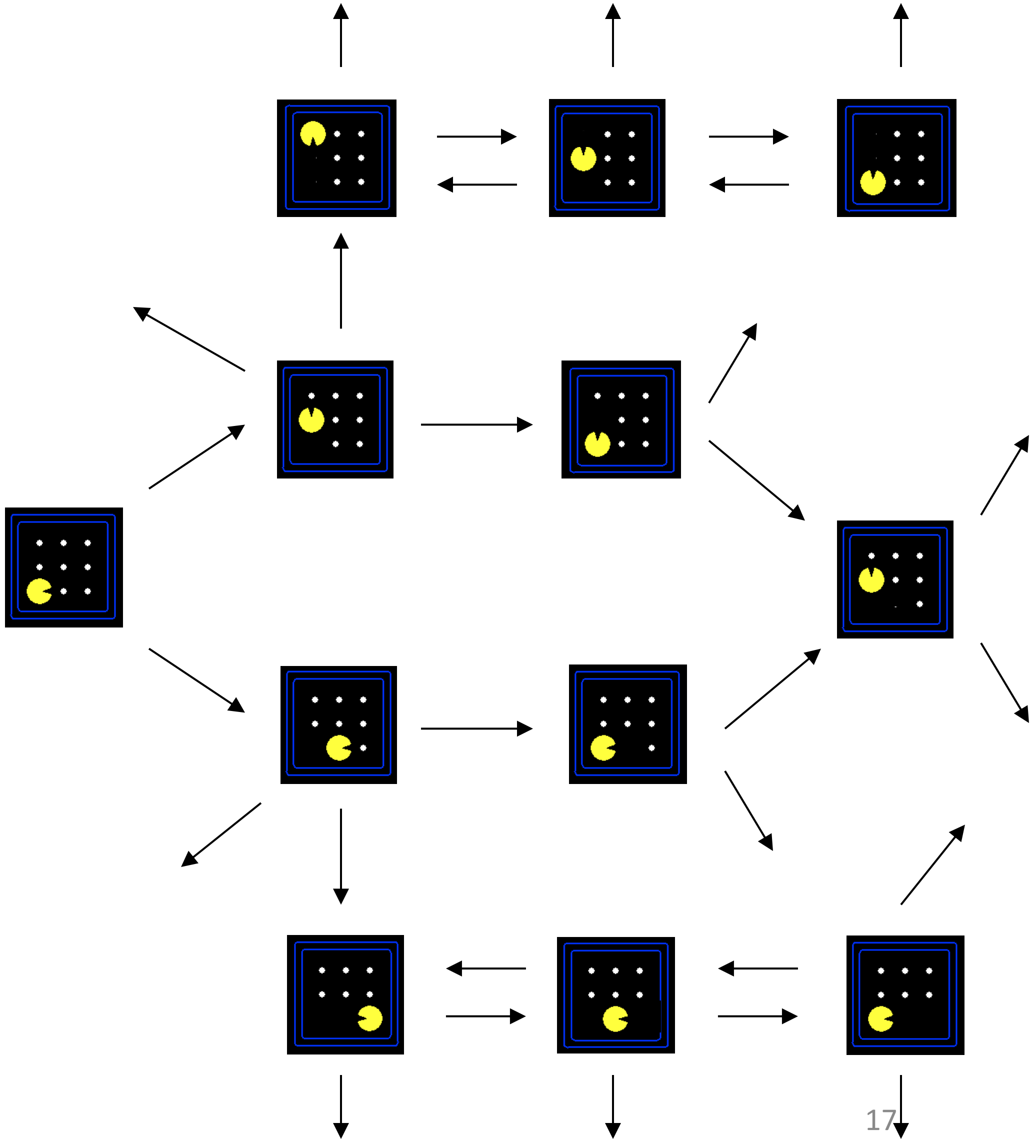


State Space Graphs

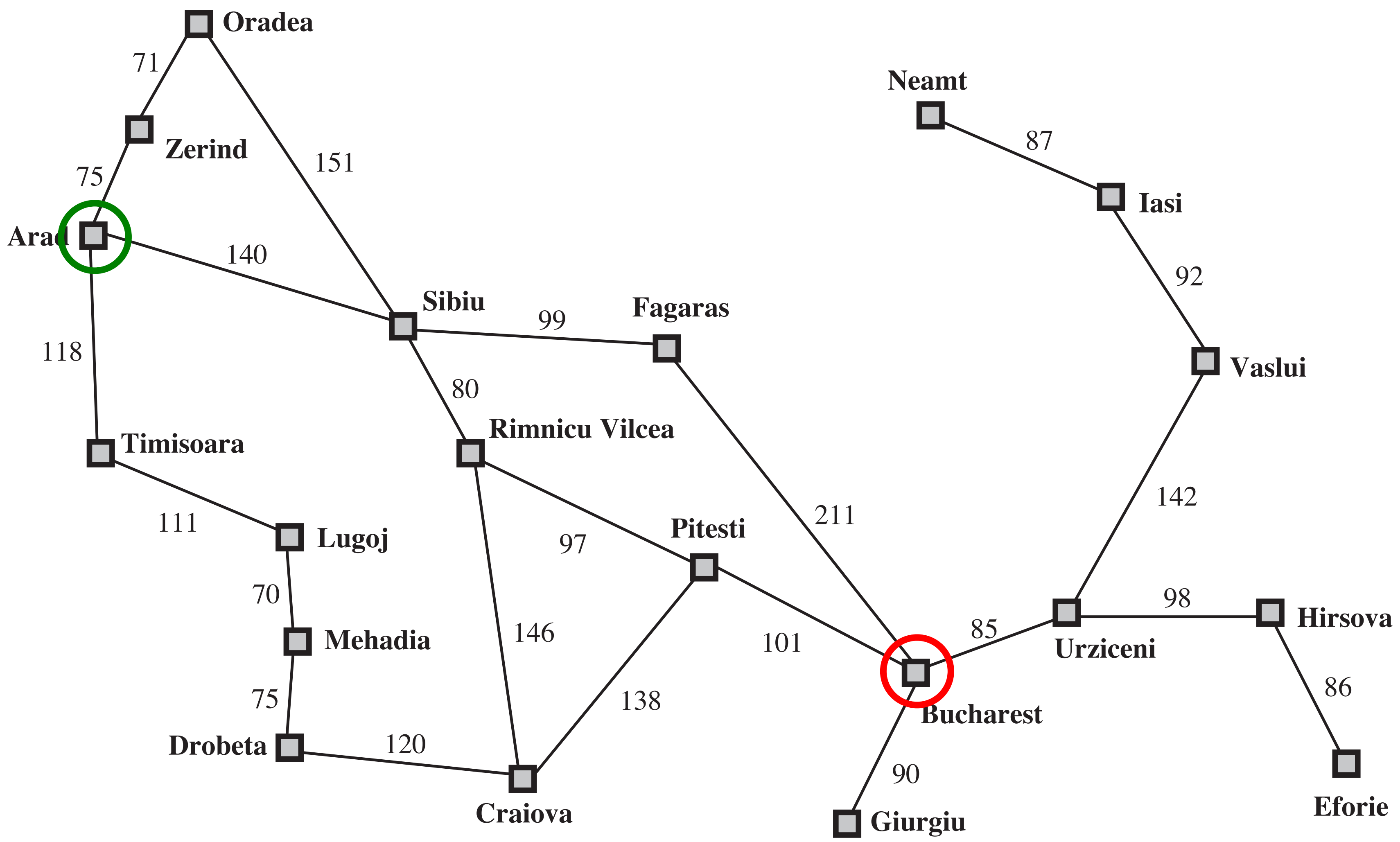
- State space graph: A mathematical representation of a search problem
  - Nodes are (abstracted) world configurations
  - Arcs represent successors (action results)
  - The goal test is a set of goal nodes (maybe only one)

- In a state space graph, each state occurs only once!

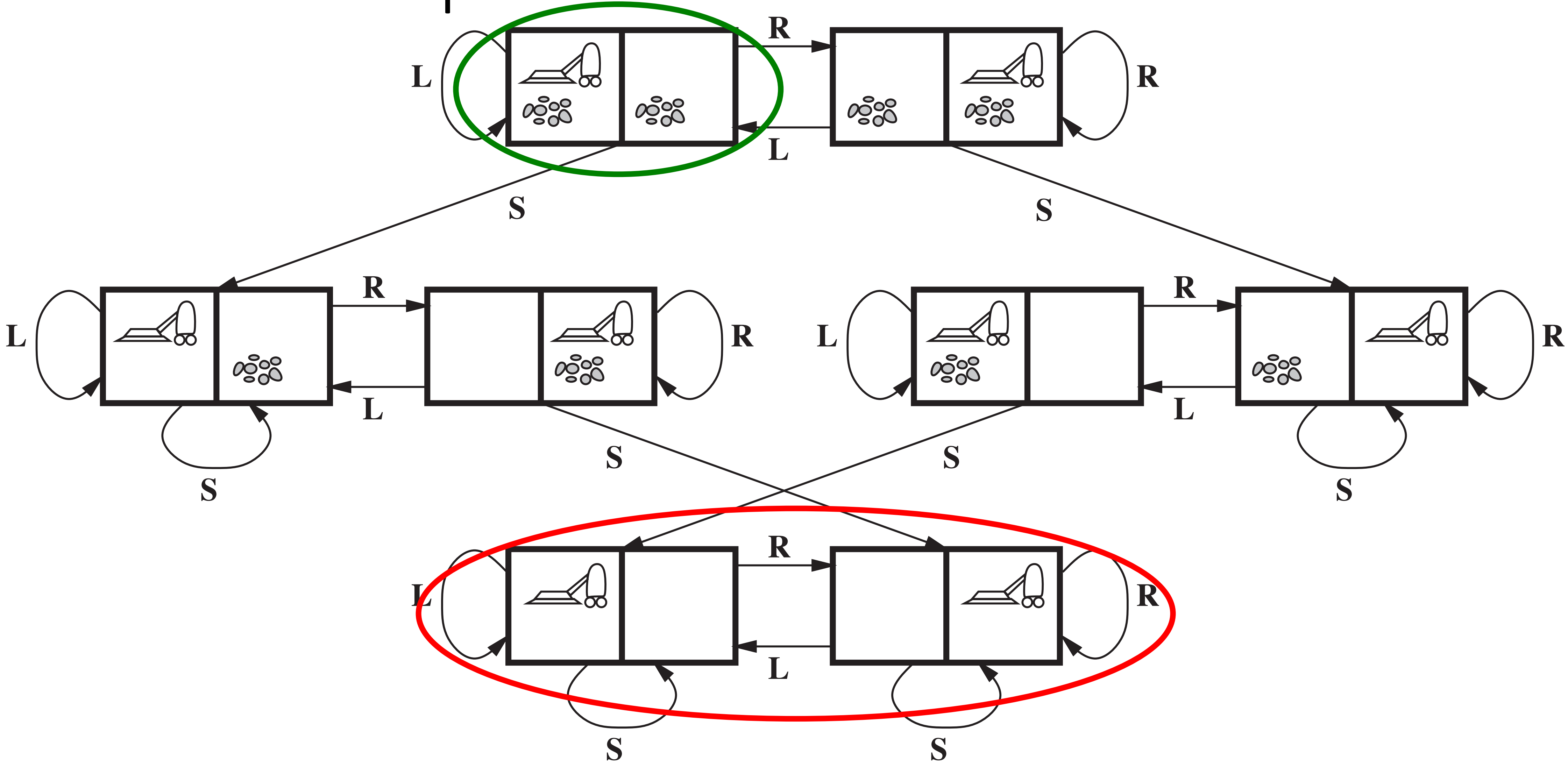
- We can rarely build this full graph in memory (it's too big), but it's a useful idea



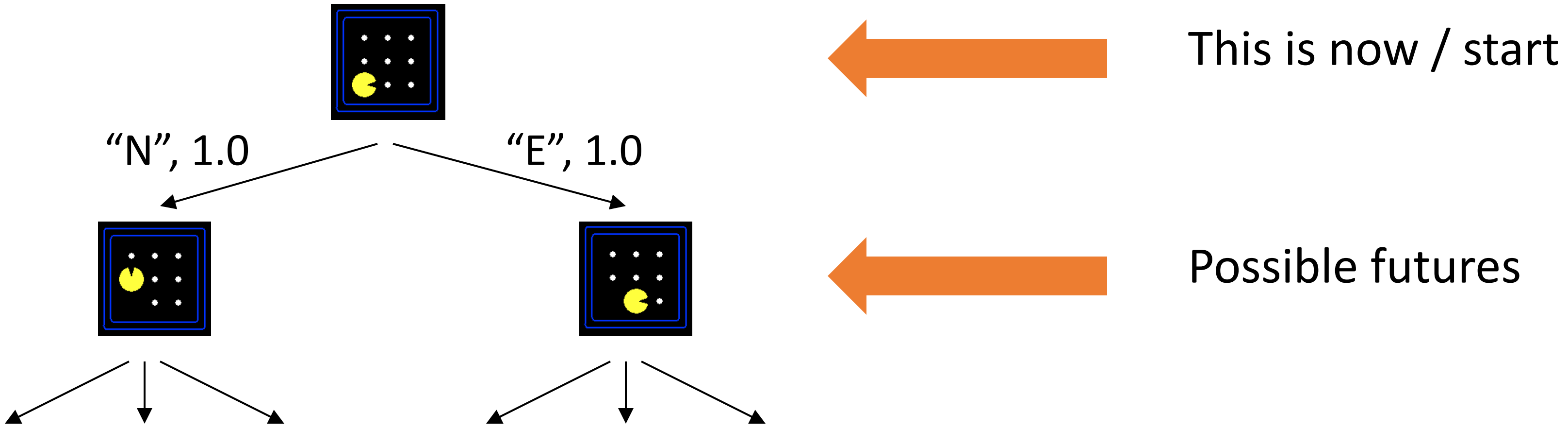
More Examples



More Examples



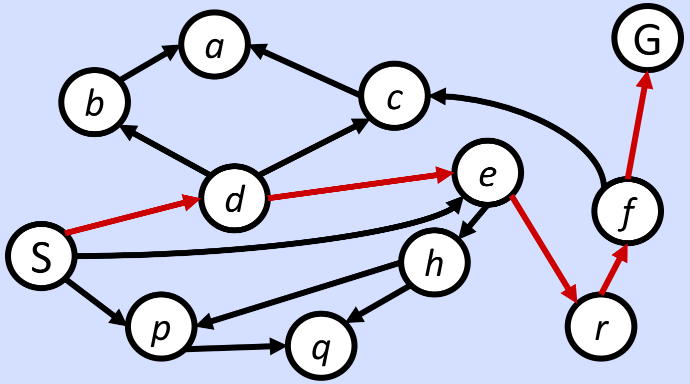
Search Trees



- A search tree:
  - A “what if” tree of plans and their outcomes
  - The start state is the root node
  - Children correspond to successors
  - Nodes show states, but correspond to PLANS that achieve those states
  - For most problems, we can never actually build the whole tree

State Space Graphs vs. Search Trees

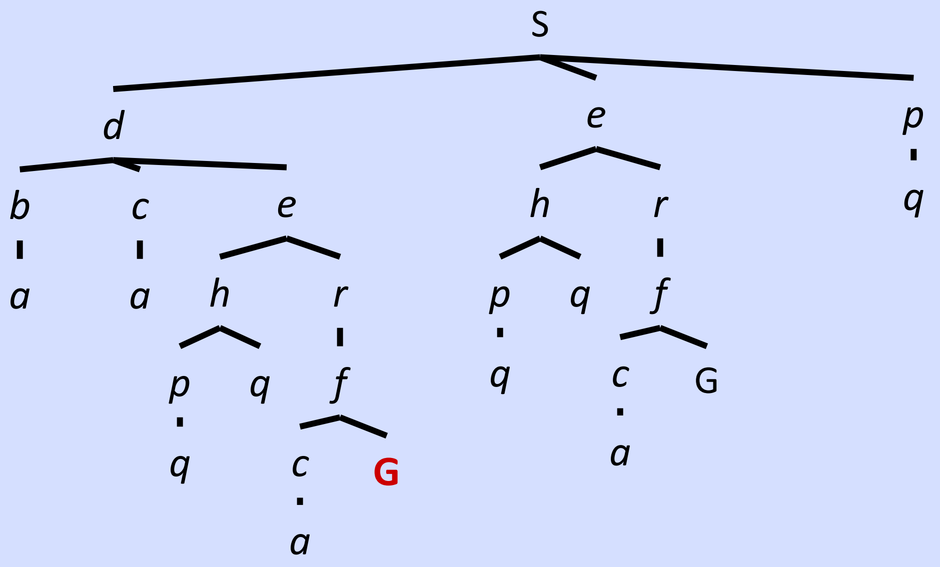
State Space Graph



Each NODE in in the search tree is an entire PATH in the state space graph

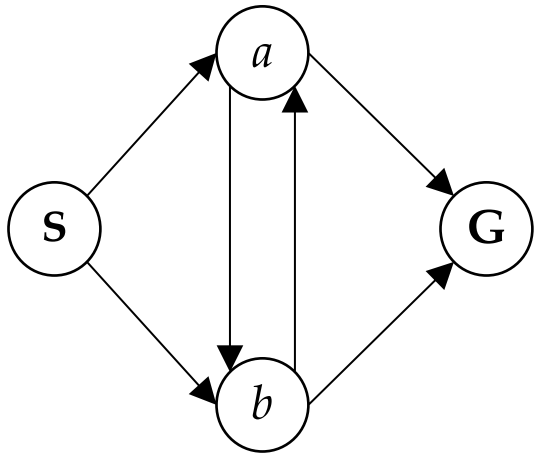
We construct both on demand – and we construct as little as possible

Search Tree

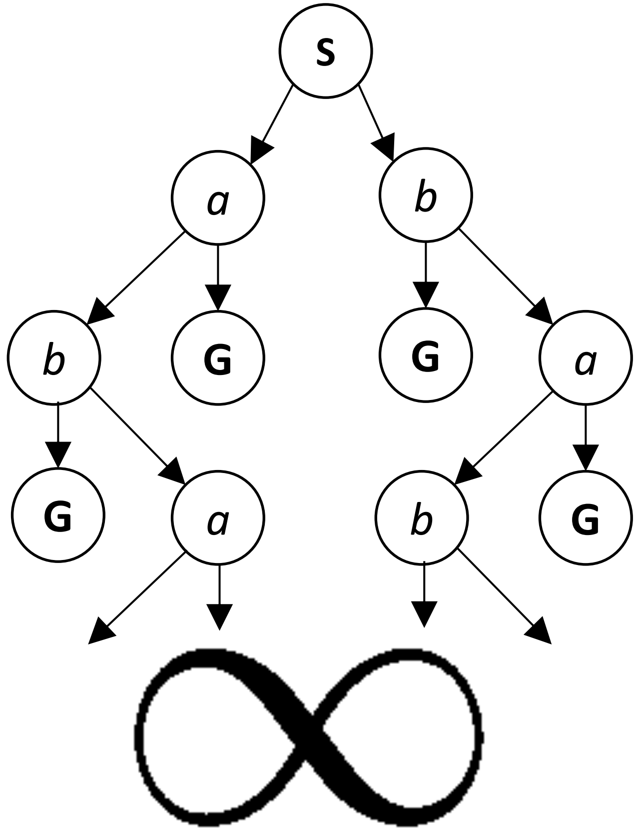


State Space Graphs vs. Search Trees

Consider this 4-state graph:

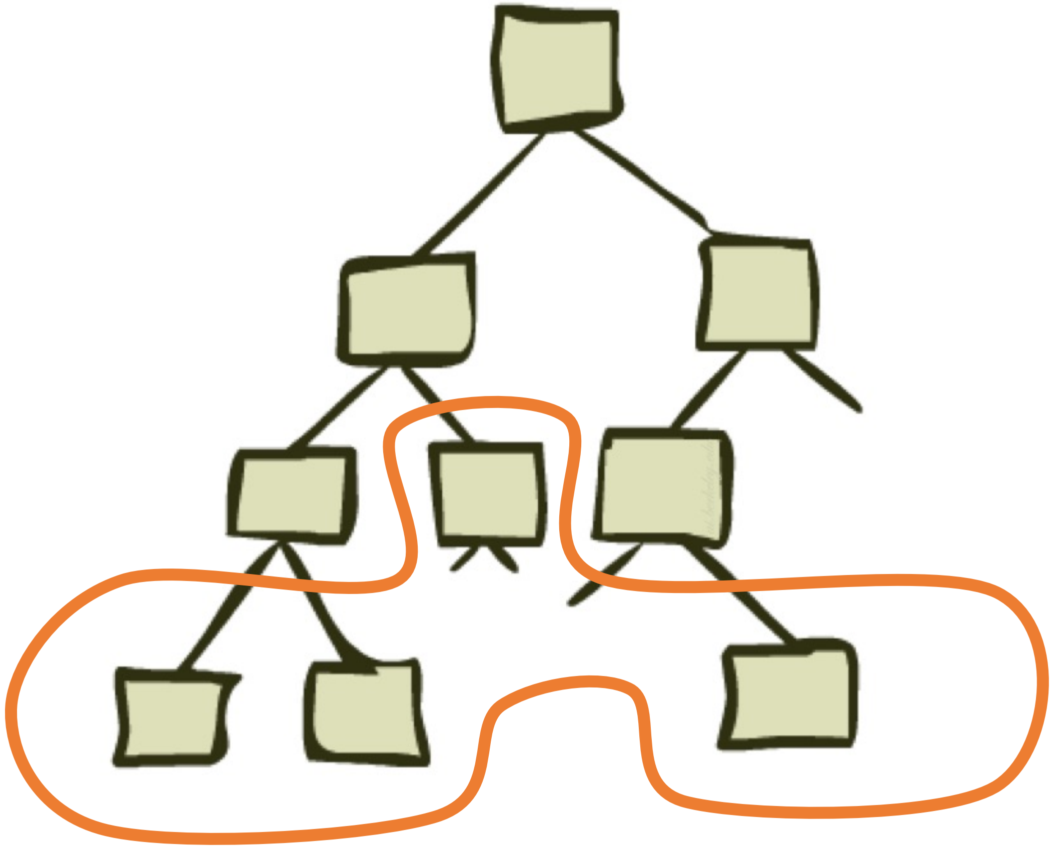


How big is its search tree (from S)?

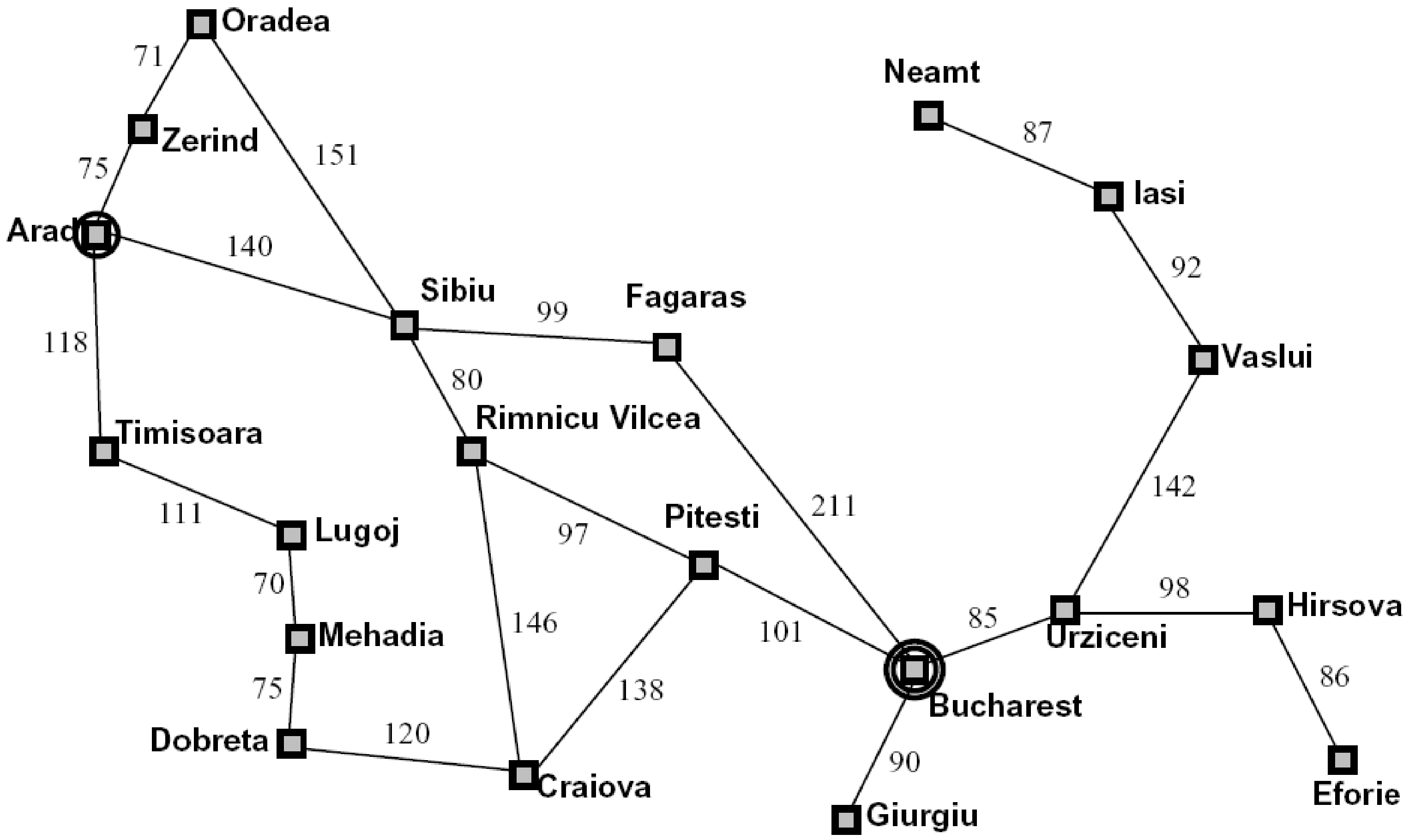


Important: Lots of repeated structure in the search tree!

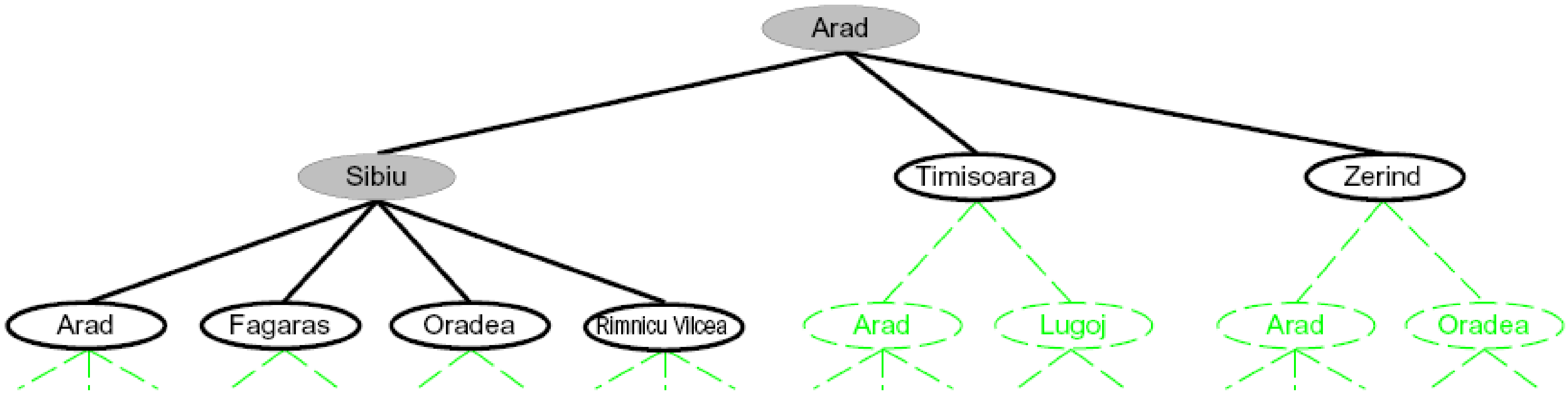
Tree Search



Search Example: Romania

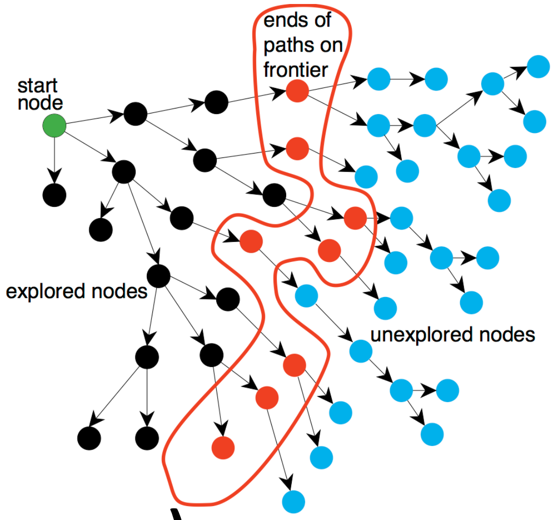


Searching with a Search Tree



- Search:
  - Expand out potential plans (tree nodes)
  - Maintain a fringe of partial plans under consideration
  - Try to expand as few tree nodes as possible

General Tree Search



function TREE_SEARCH(problem) returns a solution, or failure

initialize the frontier as **a specific work list (stack, queue, priority queue)**

add initial state of problem to frontier

loop do

if the frontier is empty then

return failure

choose a node and remove it from the frontier

if the node contains a goal state then

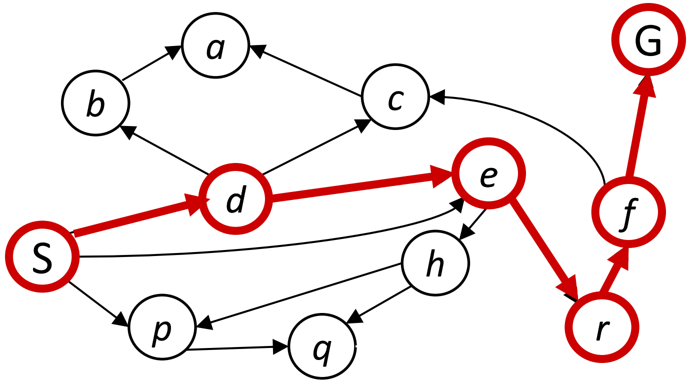
return the corresponding solution

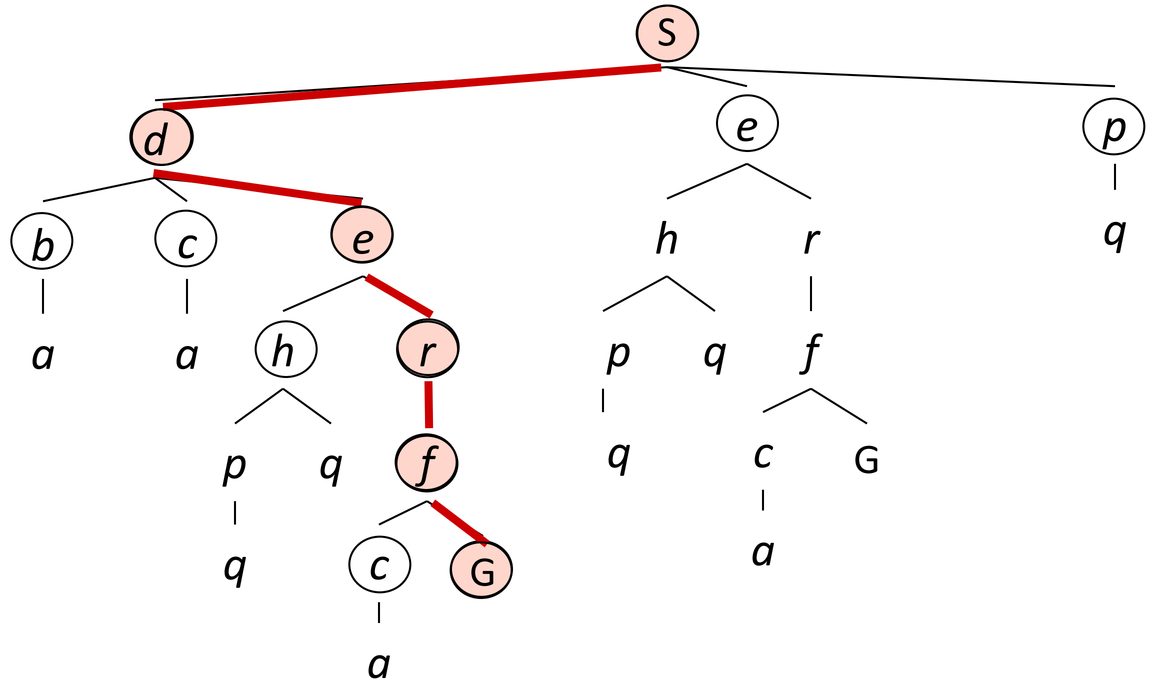
for each resulting child from node

add child to the frontier

- Important ideas:
  - Fringe
  - Expansion
  - Exploration strategy
- Main question
  - which fringe nodes to explore?

Example: Tree Search



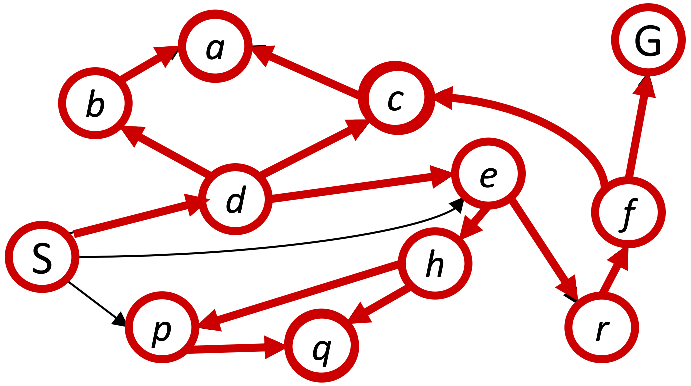


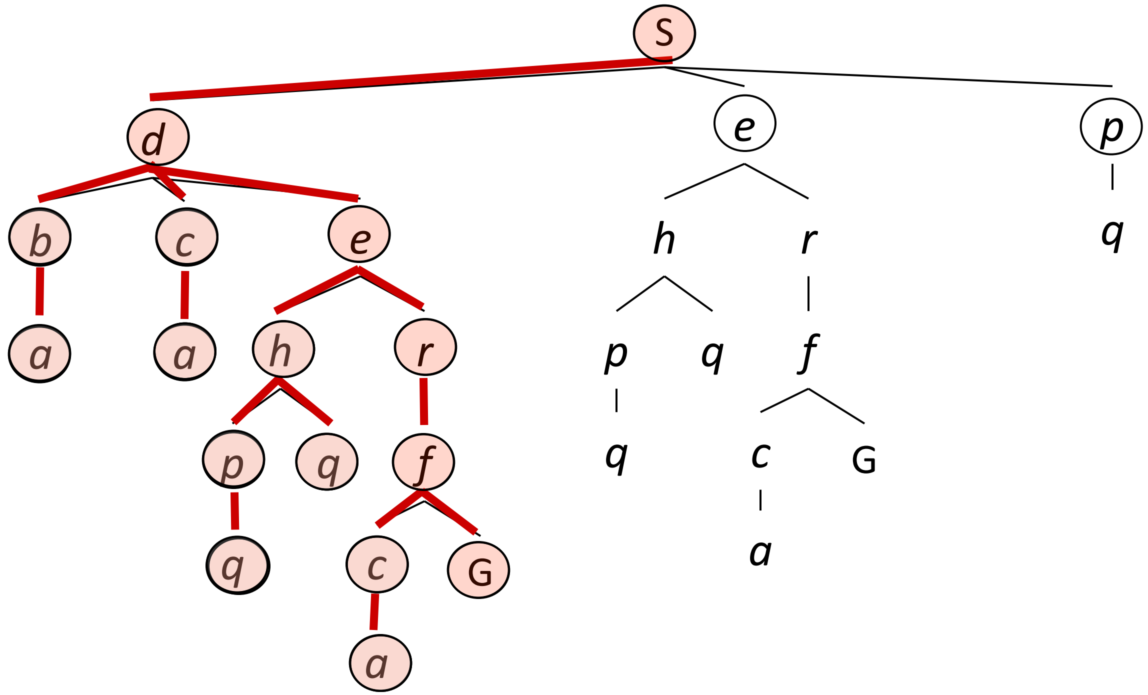
~~s~~
~~s → d~~
s → e
s → p
s → d → b
s → d → c
~~s → d → e~~
s → d → e → h
~~s → d → e → r~~
~~s → d → e → r → f~~
s → d → e → r → f → c
~~s → d → e → r → f → G~~

Depth-First (Tree) Search

Strategy: expand a deepest node first

Implementation: Fringe is a LIFO stack

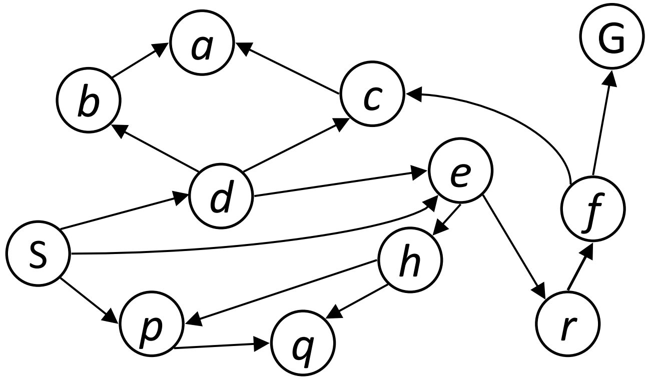




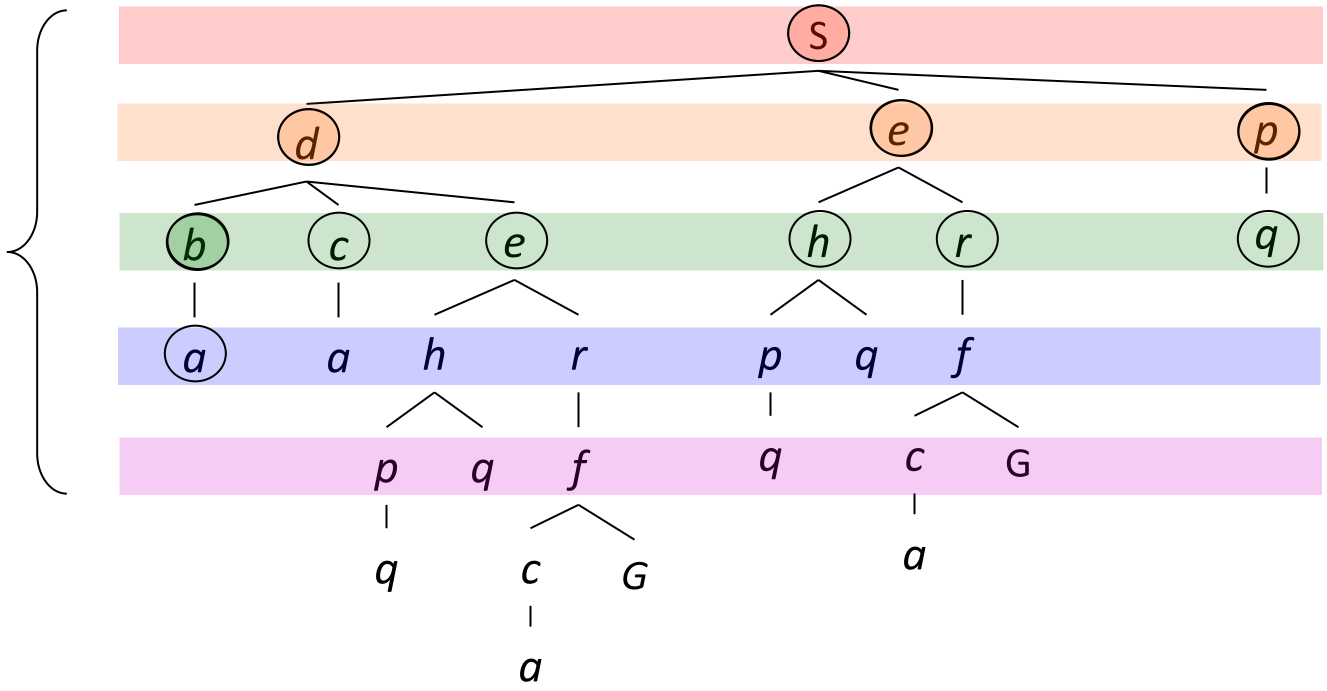
Breadth-First (Tree) Search

Strategy: expand a shallowest node first

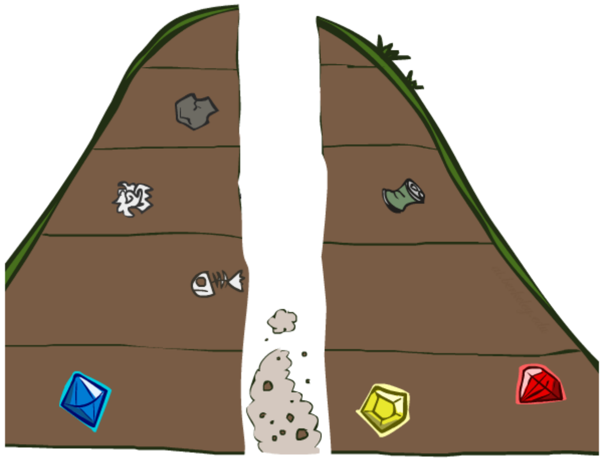
Implementation: Fringe is a FIFO queue



Search Tiers

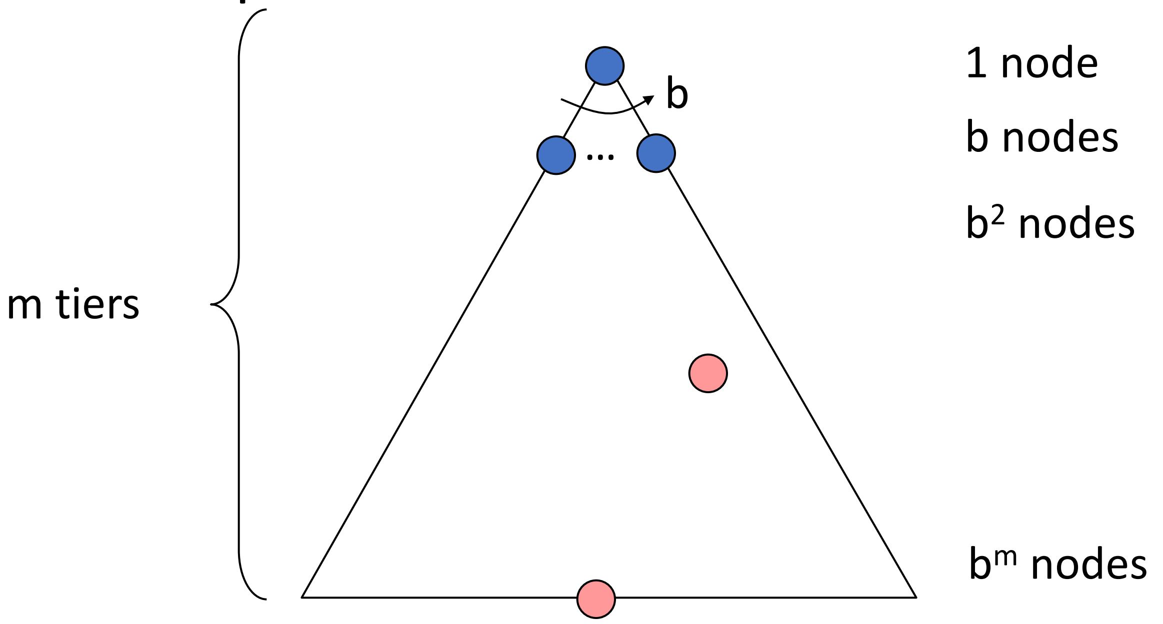


Search Algorithm Properties



- Complete: Guaranteed to find a solution if one exists?
- Optimal: Guaranteed to find the least cost path?
- Time complexity?
- Space complexity?

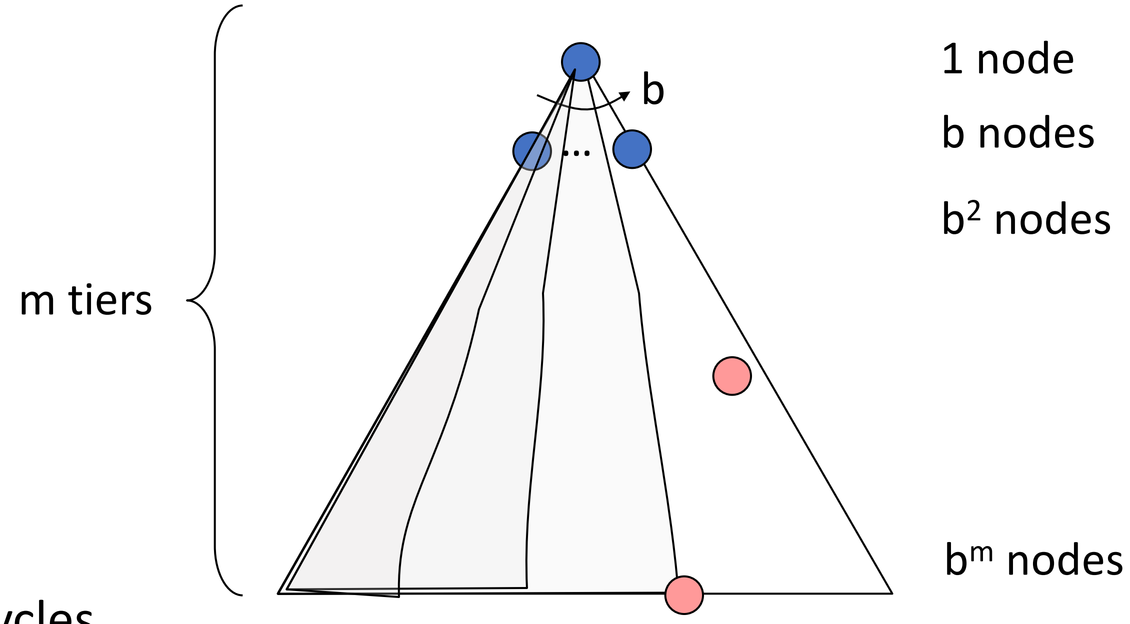
- Cartoon of search tree:
  - b is the branching factor
  - m is the maximum depth
  - solutions at various depths



- Number of nodes in entire tree?
  - $1 + b + b^2 + \ldots + b^m = O(b^m)$

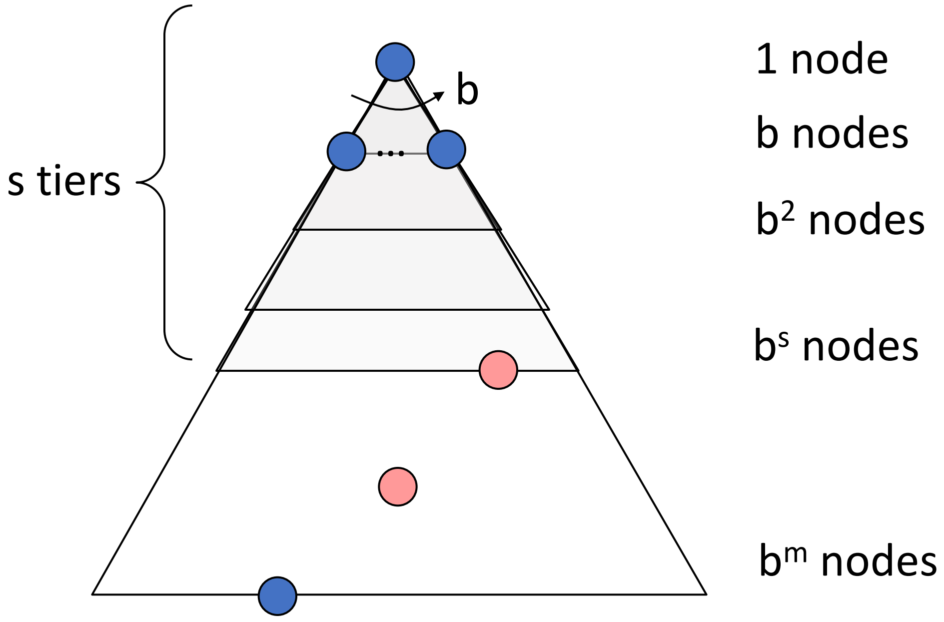
Depth-First Search (DFS) Properties

- What nodes DFS expand?
  - Some left prefix of the tree.
  - Could process the whole tree!
  - If m is finite, takes time $O(b^m)$
- How much space does the fringe take?
  - Only has siblings on path to root, so O(bm)
- Is it complete?
  - m could be infinite, so only if we prevent cycles (more later)
- Is it optimal?
  - No, it finds the “leftmost” solution, regardless of depth or cost

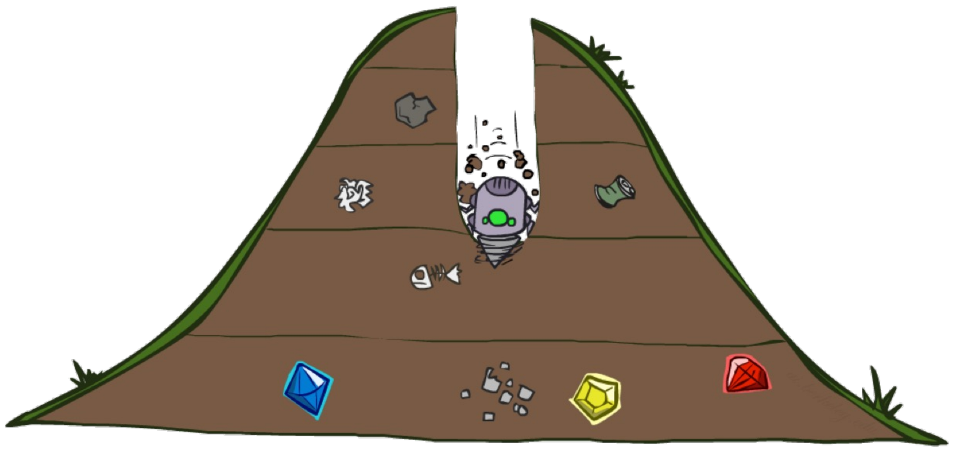


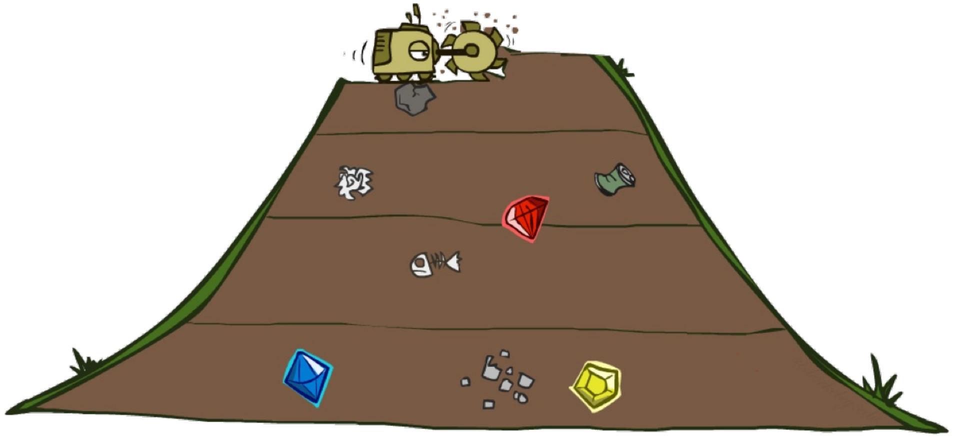
Breadth-First Search (BFS) Properties

- What nodes does BFS expand?
  - Processes all nodes above shallowest solution
  - Let depth of shallowest solution be s
  - Search takes time $O(b^s)$
- How much space does the fringe take?
  - Has roughly the last tier, so $O(b^s)$
- Is it complete?
  - s must be finite if a solution exists
- Is it optimal?
  - Only if costs are all 1 (more on costs later)



DFS vs BFS





- When will BFS outperform DFS?
- When will DFS outperform BFS?

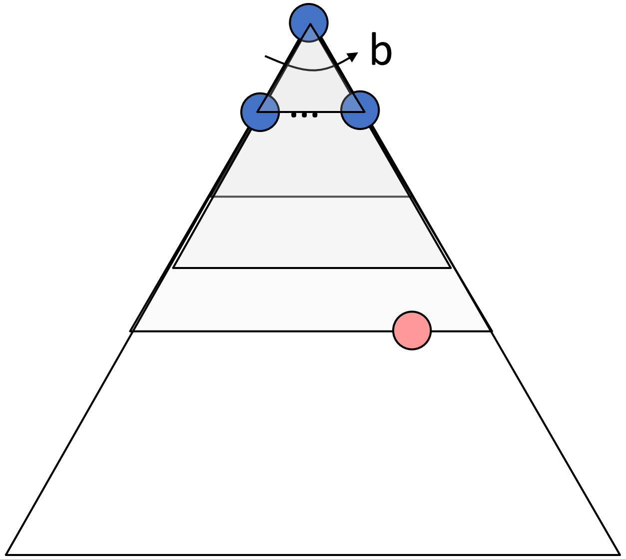
[Demo: dfs/bfs maze water (L2D6)]

Video of Demo Maze Water DFS/BFS

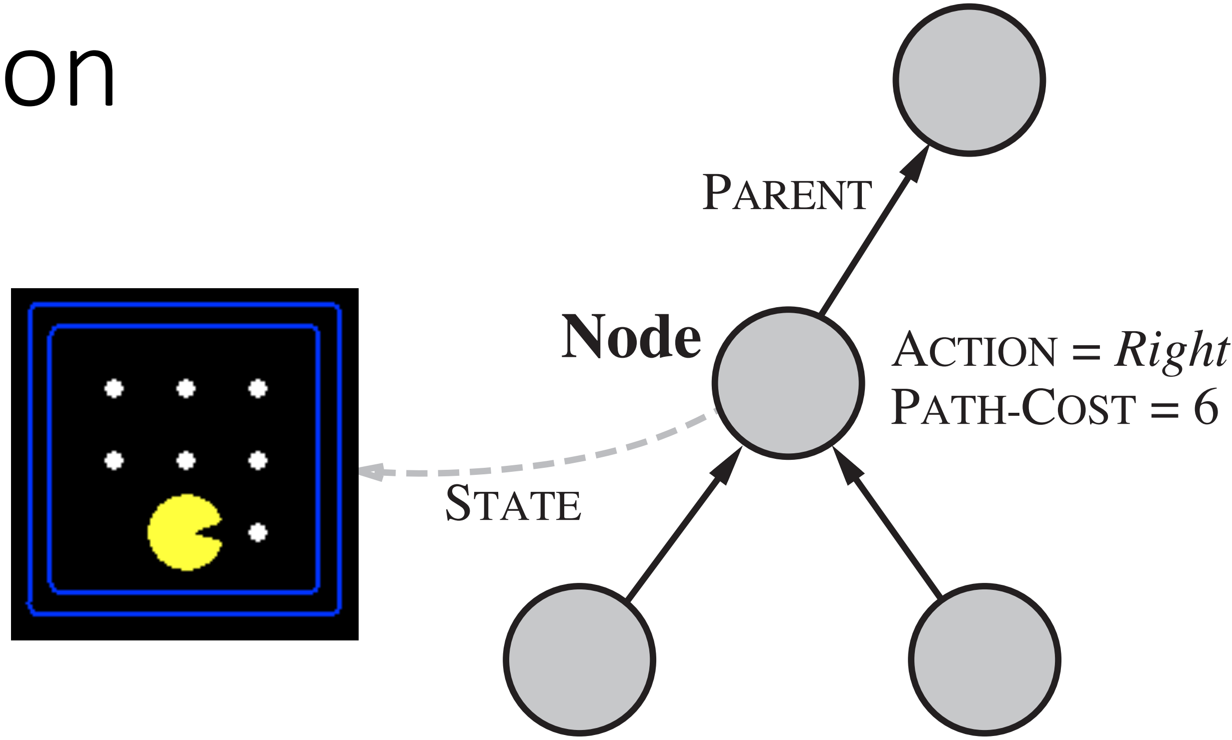
Iterative Deepening

- Idea: get DFS's space advantage with BFS's time / shallow-solution advantages
  - Run a DFS with depth limit 1. If no solution…
  - Run a DFS with depth limit 2. If no solution…
  - Run a DFS with depth limit 3. …..

- Isn't that wastefully redundant?
  - Generally most work happens in the lowest level searched, so not so bad!

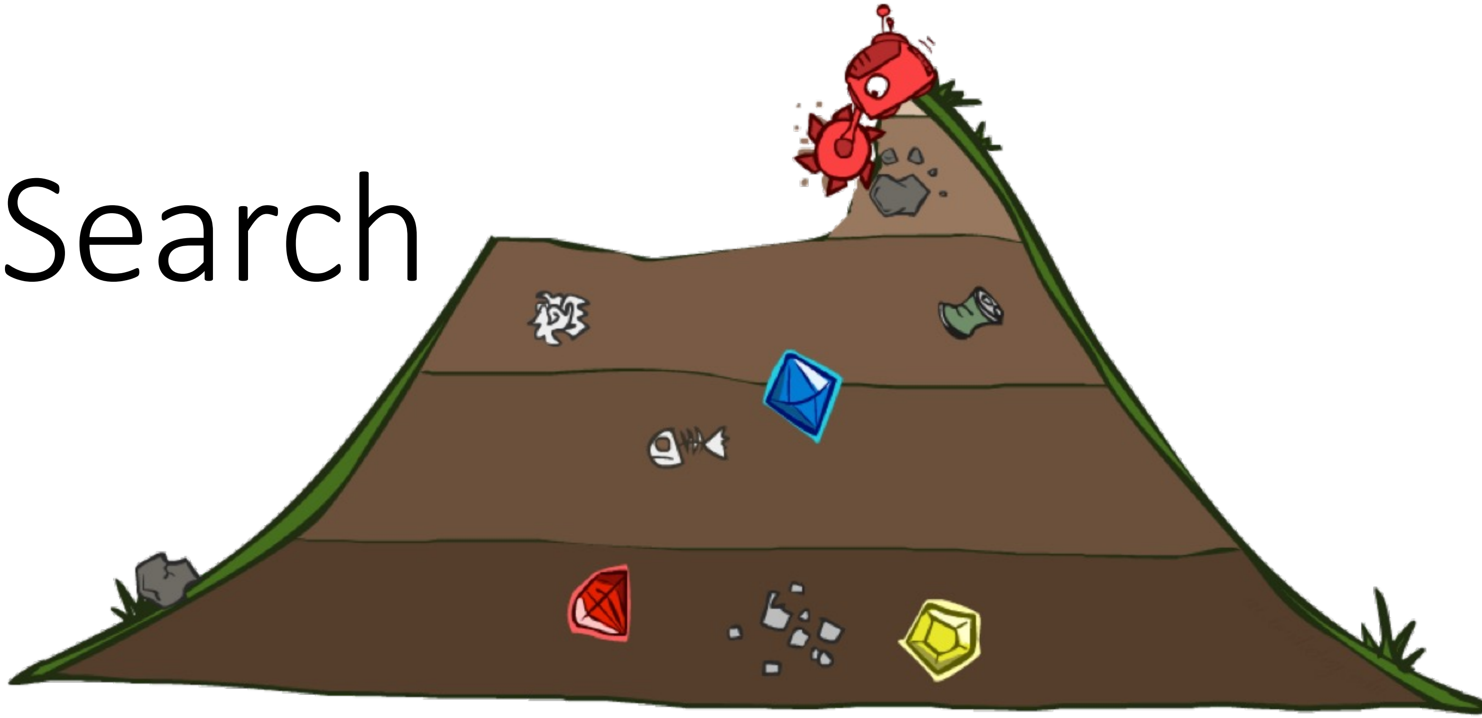


A Note on Implementation



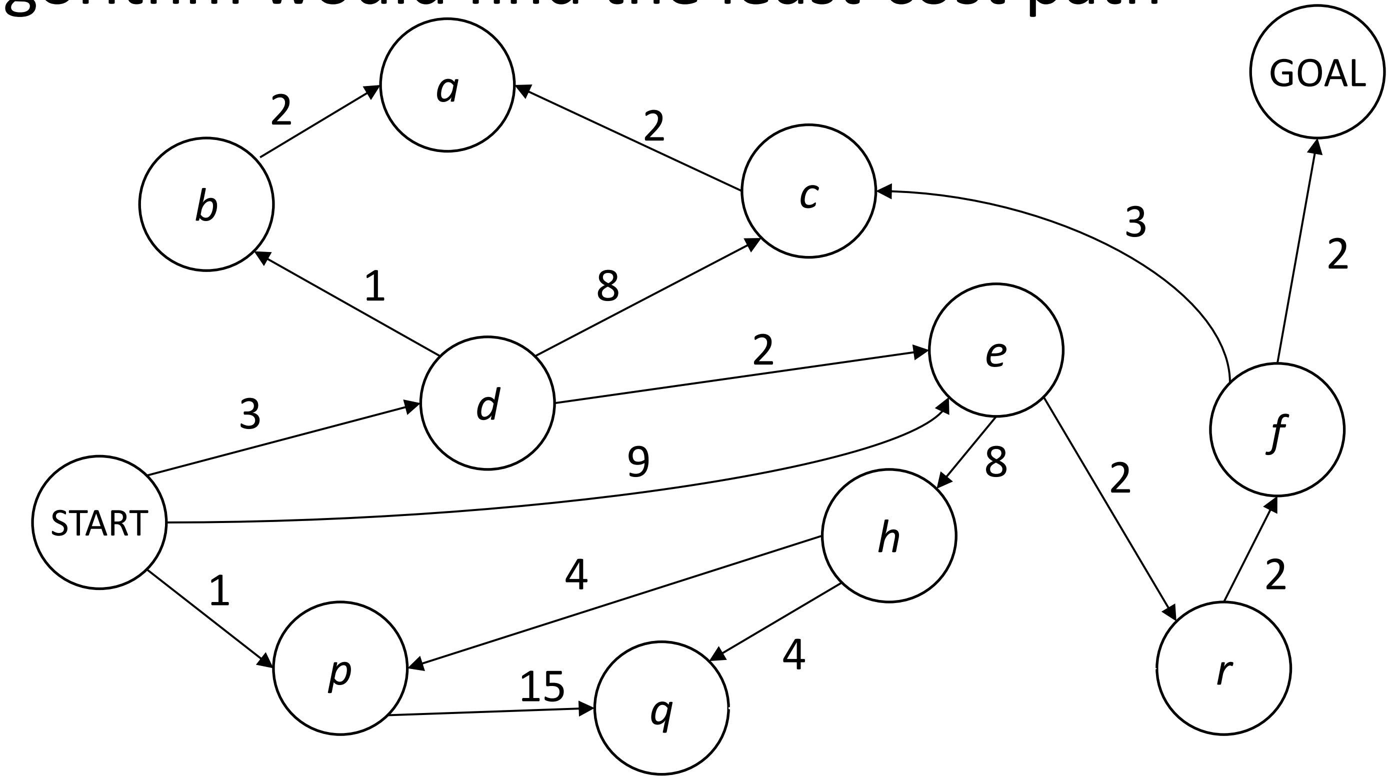
- Nodes have
  - state, parent, action, path-cost
- A child of node by action a has
  - state = Transition(node.state, a)
  - parent = node
  - action = a
  - path-cost = node.path_cost + step_cost(node.state, a, self.state)
- Extract solution by tracing back parent pointers, collecting actions

Uniform Cost Search



Finding a Least-Cost Path

- BFS finds the shortest path in terms of number of actions, but not the least-cost path
- A similar algorithm would find the least-cost path

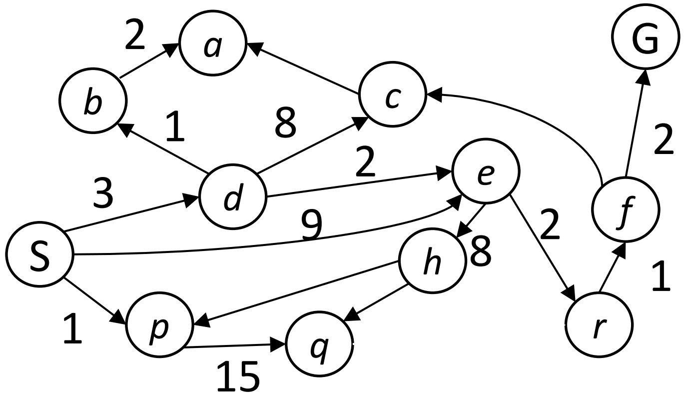


How?

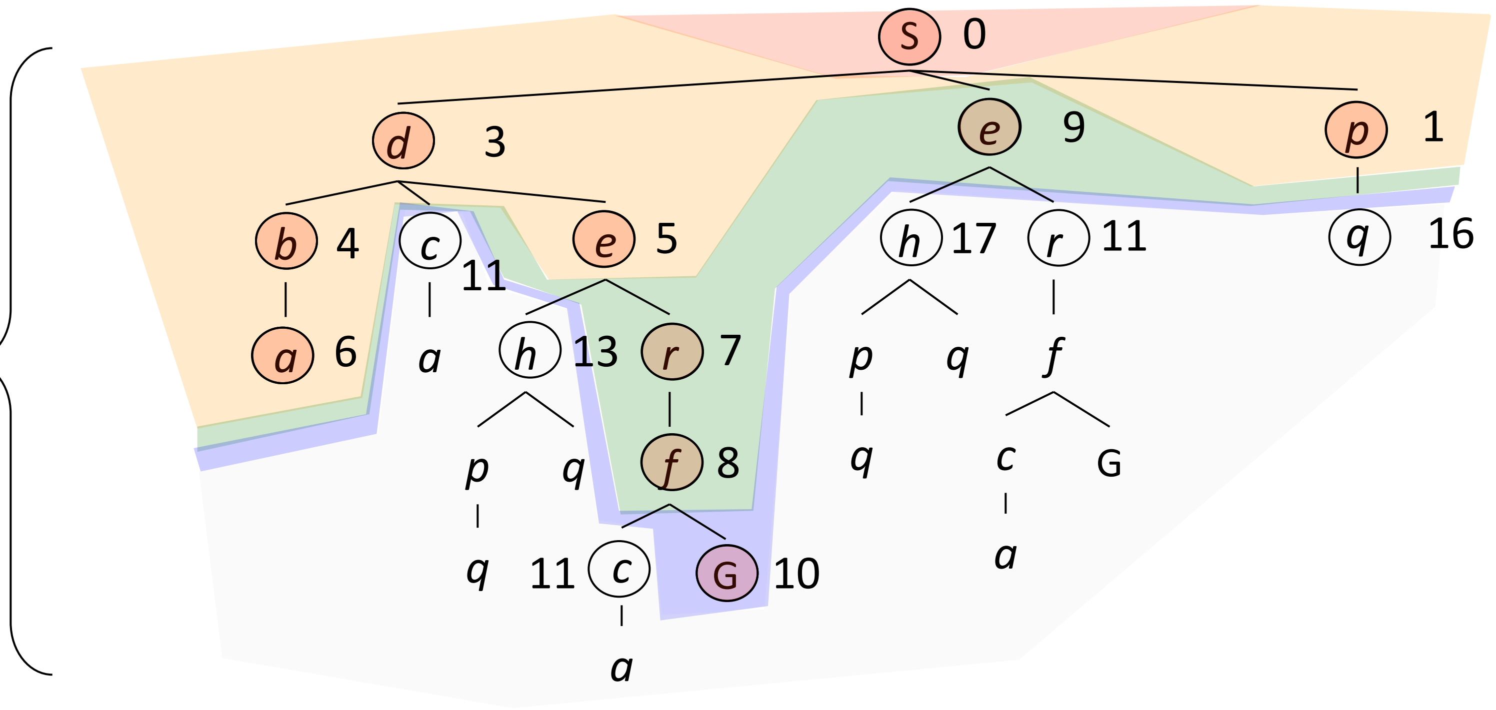
Uniform Cost Search

Strategy: expand a cheapest node first:

Fringe is a priority queue (priority: cumulative cost)



Cost contours

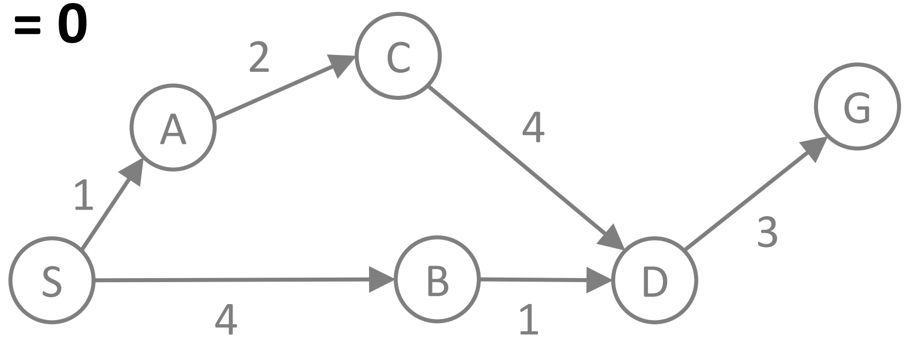


Uniform Cost Search 2

function UNIFORM-COST-SEARCH(problem) returns a solution, or failure

initialize the frontier as a **priority queue using node's path_cost as the priority**

add initial state of problem to frontier **with path_cost = 0**



loop do

if the frontier is empty then

return failure

choose a node (with minimal path_cost) and remove it from the frontier

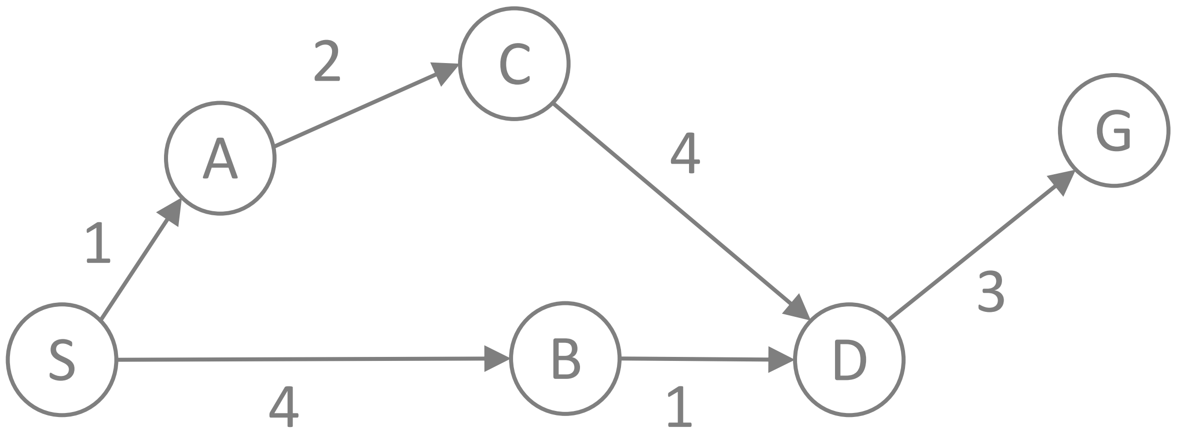
if the node contains a goal state then

return the corresponding solution

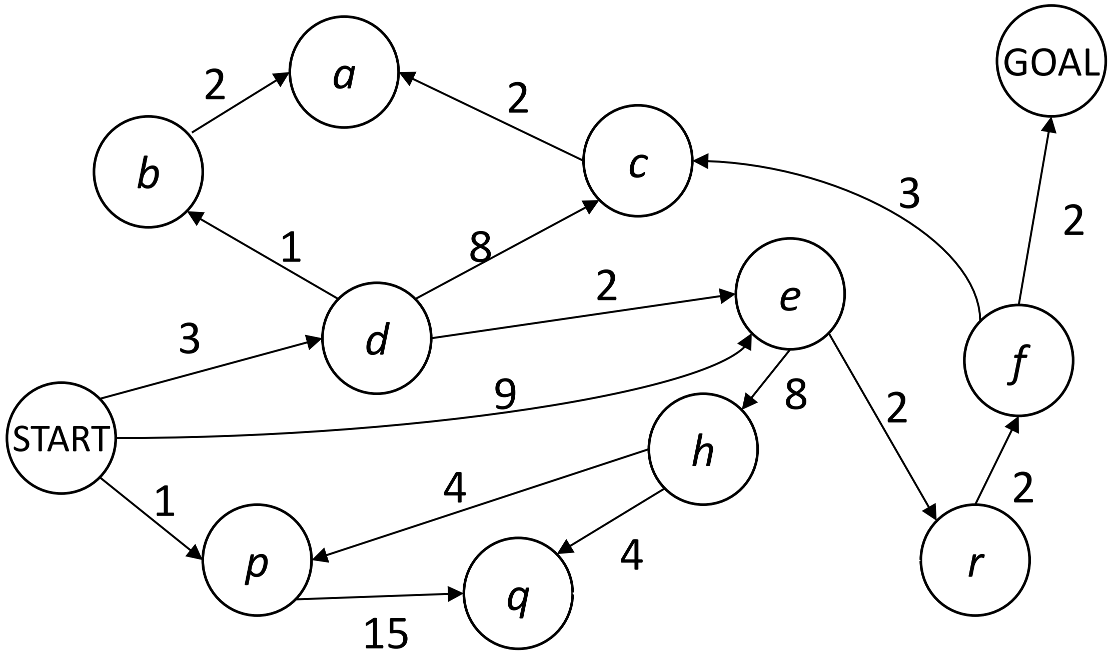
for each resulting child from node

add child to the frontier with **path_cost** = **path_cost**(node) + cost(node, child)

Walk-through UCS

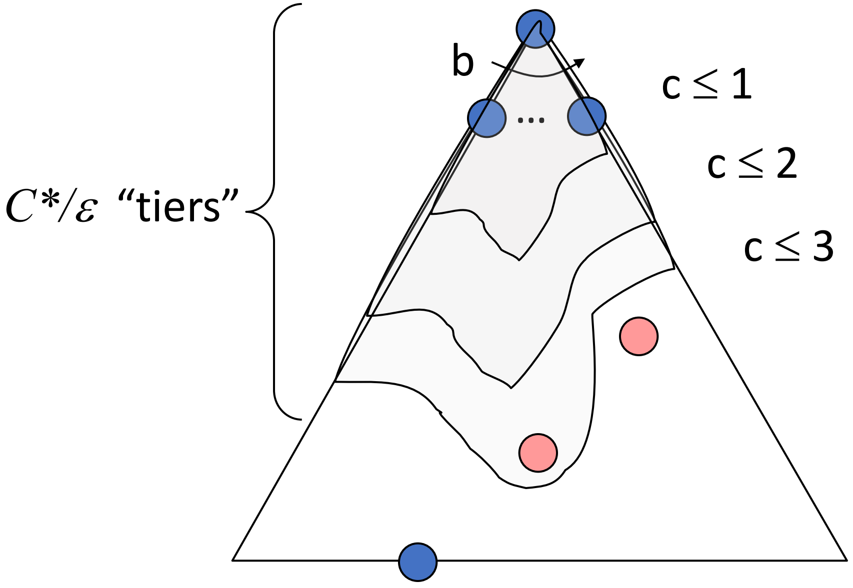


Walk-through UCS



Uniform Cost Search (UCS) Properties

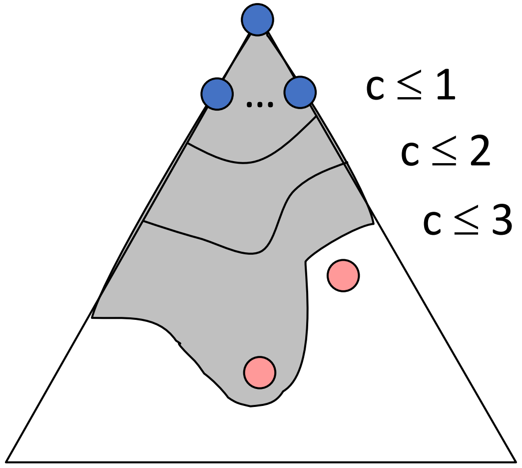
- What nodes does UCS expand?
  - Processes all nodes with cost less than cheapest solution!
  - If that solution costs C^* and arcs cost at least ε, then the "effective depth" is roughly C^*/ε
  - Takes time O($b^{C^*/\varepsilon}$) (exponential in effective depth)

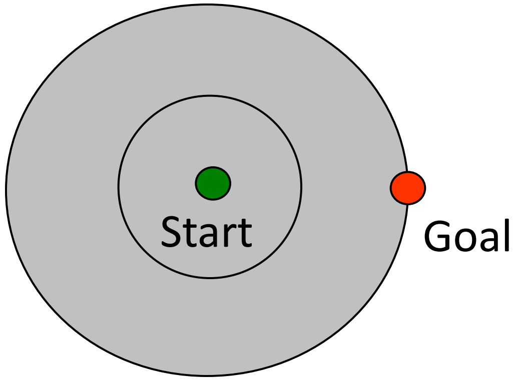


- How much space does the fringe take?
  - Has roughly the last tier, so O($b^{C^*/\varepsilon}$)
- Is it complete?
  - Assuming best solution has a finite cost and minimum arc cost is positive, yes!
- Is it optimal?
  - Yes! (Proof next via A*)

Uniform Cost Issues

- Remember: UCS explores increasing cost contours
- The good: UCS is complete and optimal!
- The bad:
  - Explores options in every “direction”
  - No information about goal location
- We’ll fix that soon!

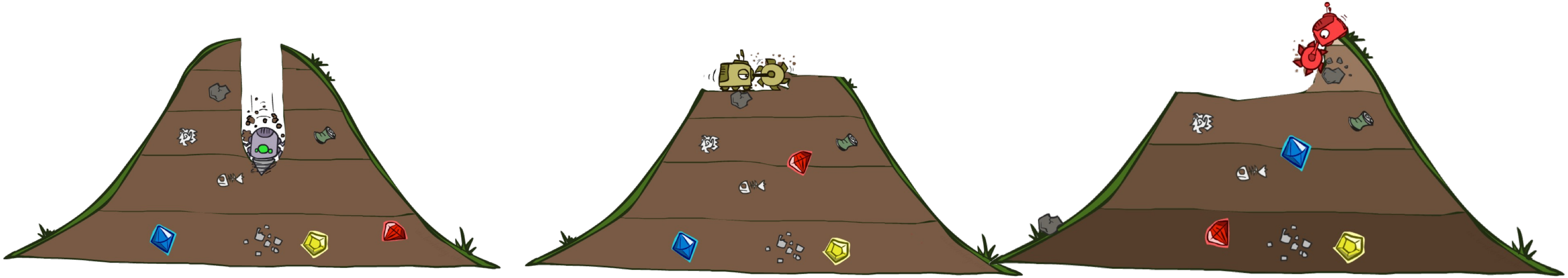




[Demo: empty grid UCS (L2D5)]
[Demo: maze with deep/shallow water DFS/BFS/UCS (L2D7)]

Video of Demo Empty UCS (same cost)

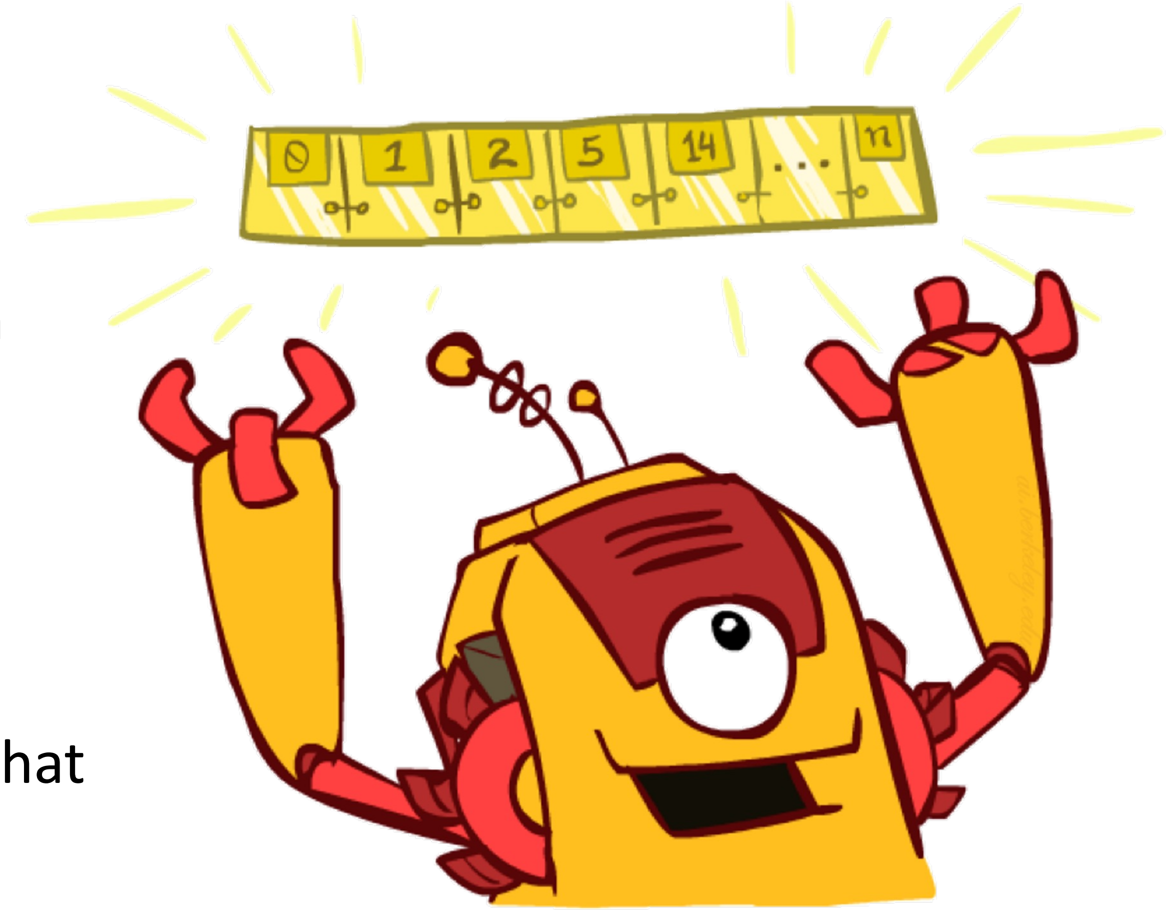
DFS, BFS, or UCS?



Video of Demo Maze with Deep/Shallow Water

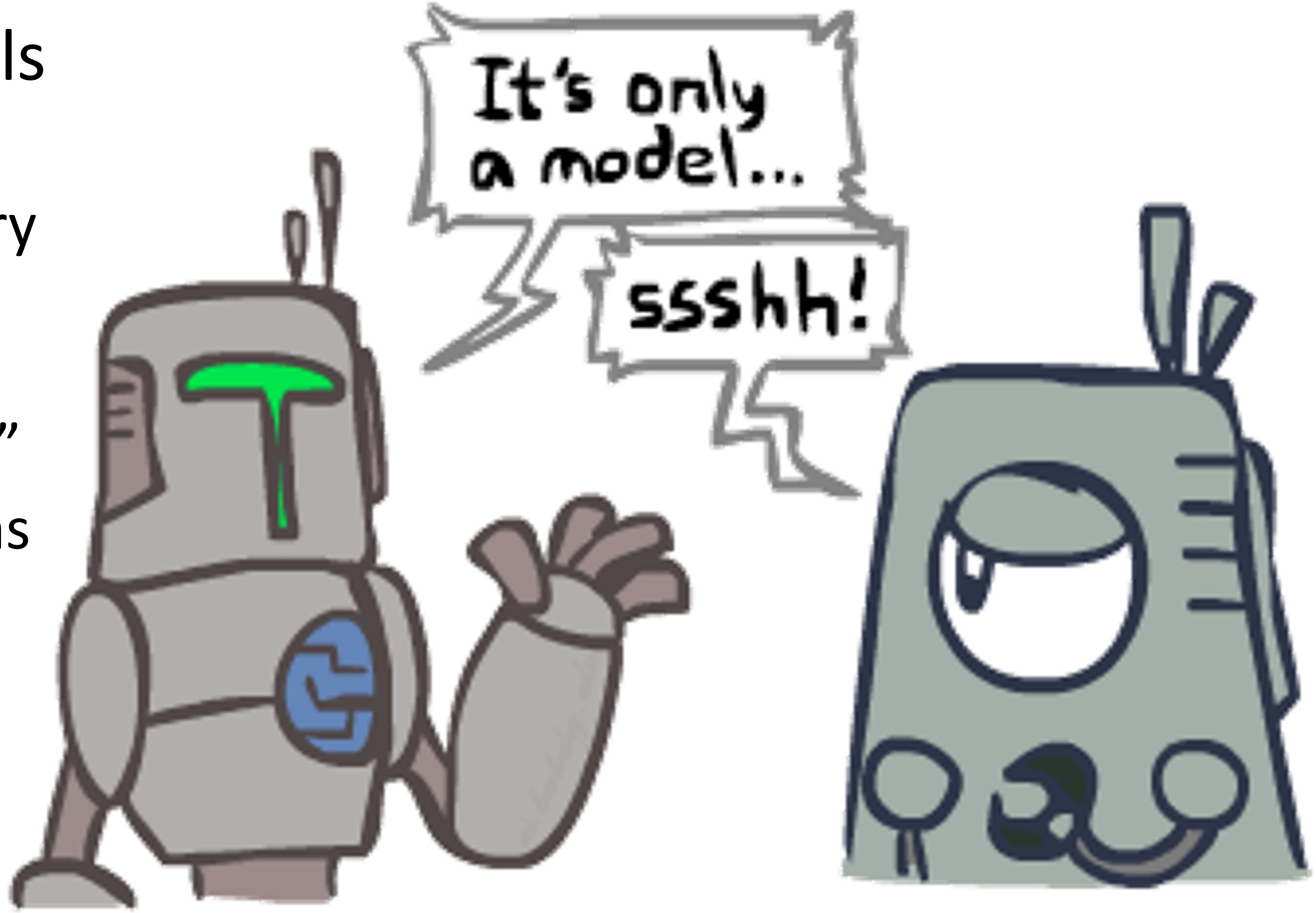
The One Queue

- All these search algorithms are the same except for fringe strategies
  - Conceptually, all fringes are priority queues (i.e. collections of nodes with attached priorities)
  - Practically, for DFS and BFS, you can avoid the log(n) overhead from an actual priority queue, by using stacks and queues
  - Can even code one implementation that takes a variable queuing object

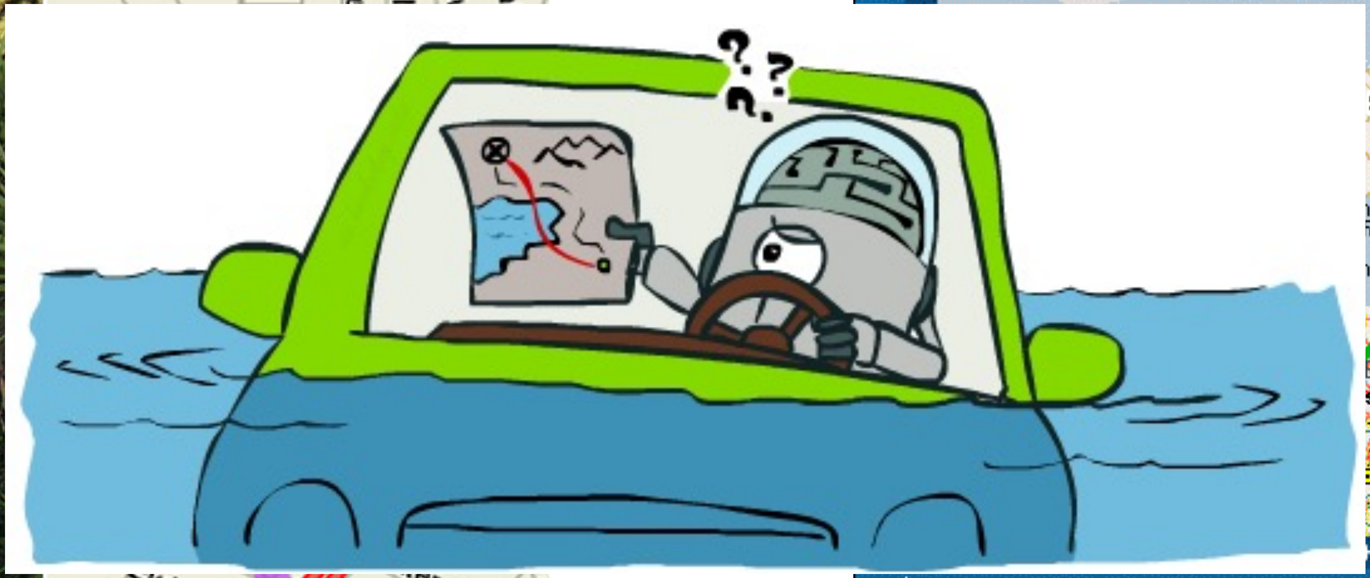


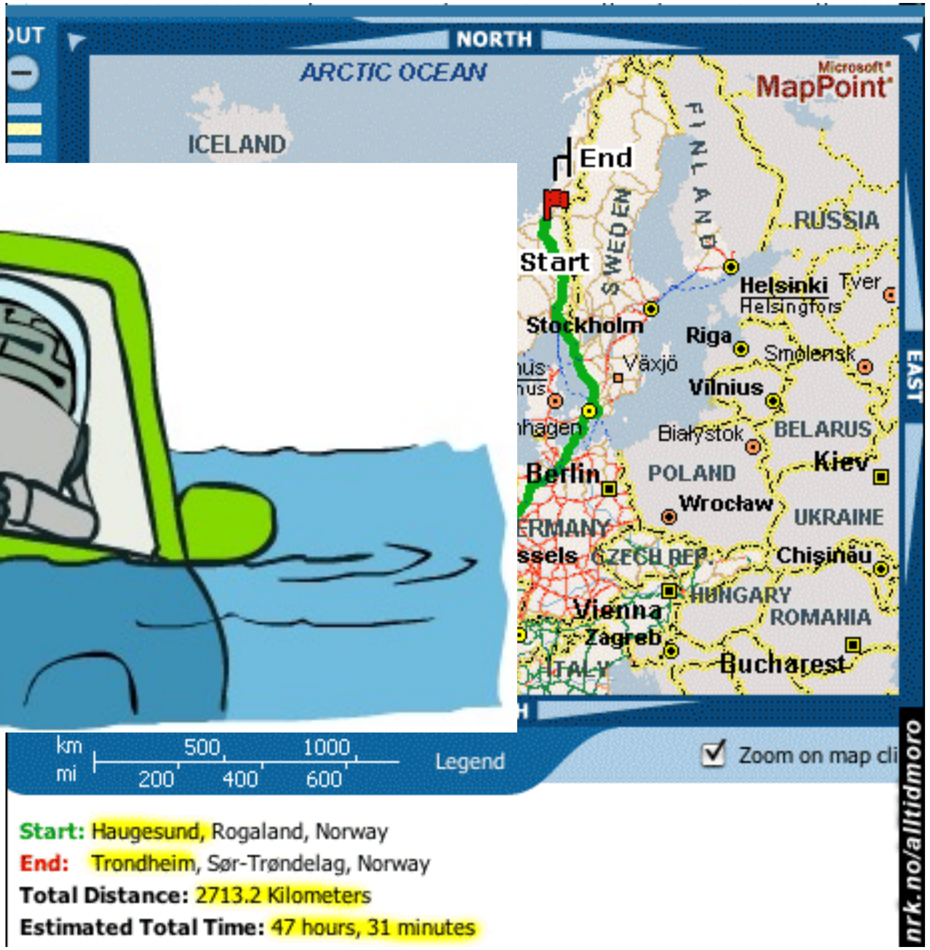
Search and Models

- Search operates over models of the world
  - The agent doesn't actually try all the plans out in the real world!
  - Planning is all "in simulation"
  - Your search is only as good as your models…



Search Gone Wrong?

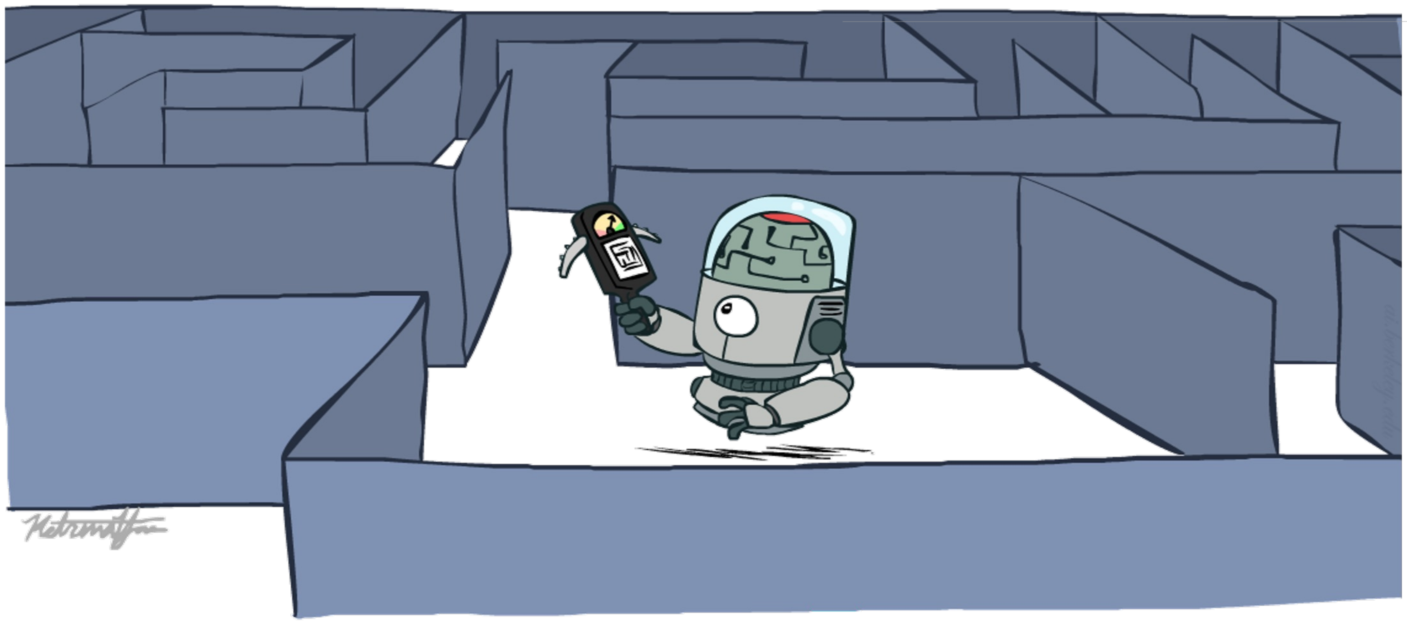




Informed Search

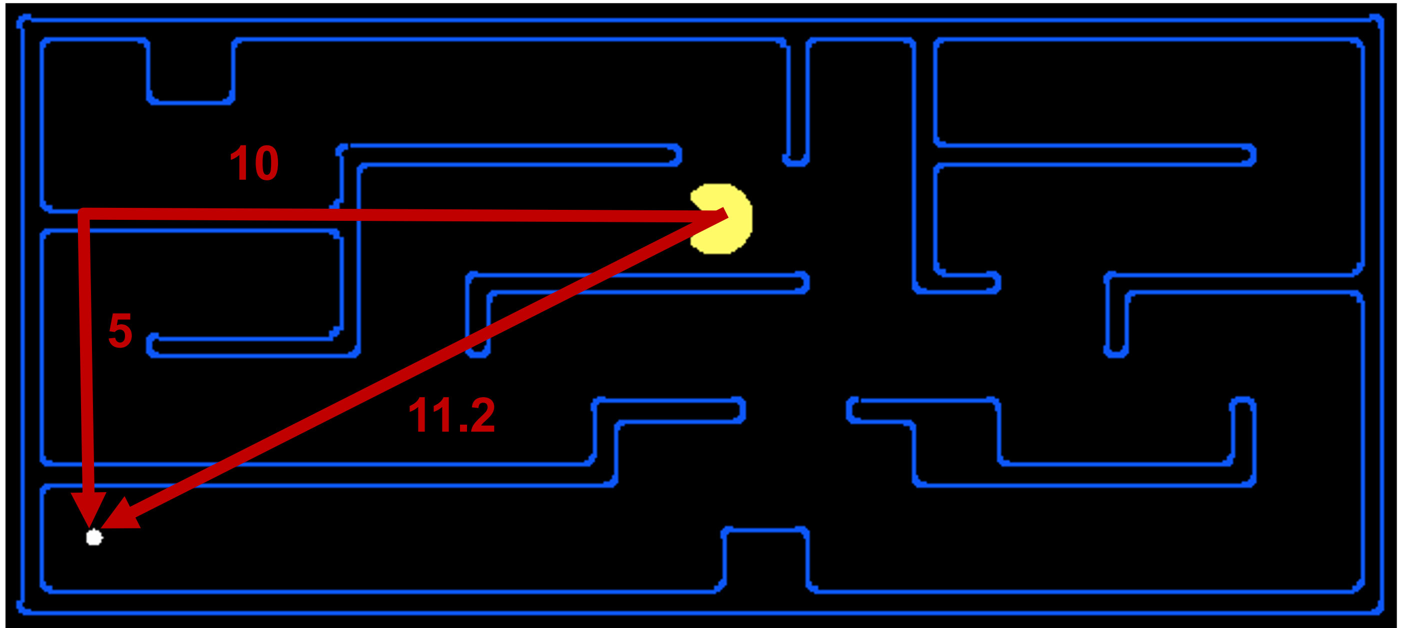
Informed Search

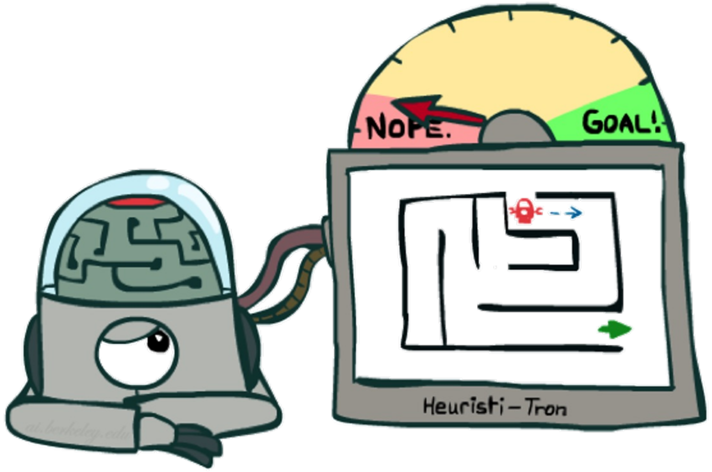
- Uninformed Search
  - DFS
  - BFS
  - UCS
- Informed Search
  - Heuristics
  - Greedy Search
  - A* Search
  - Graph Search

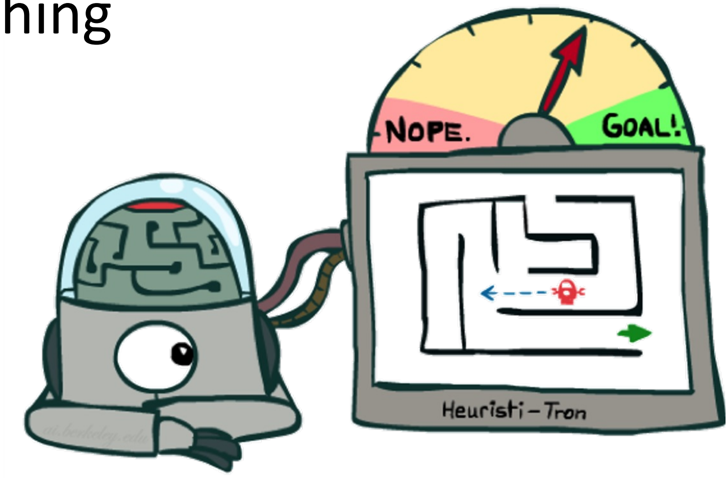


Search Heuristics

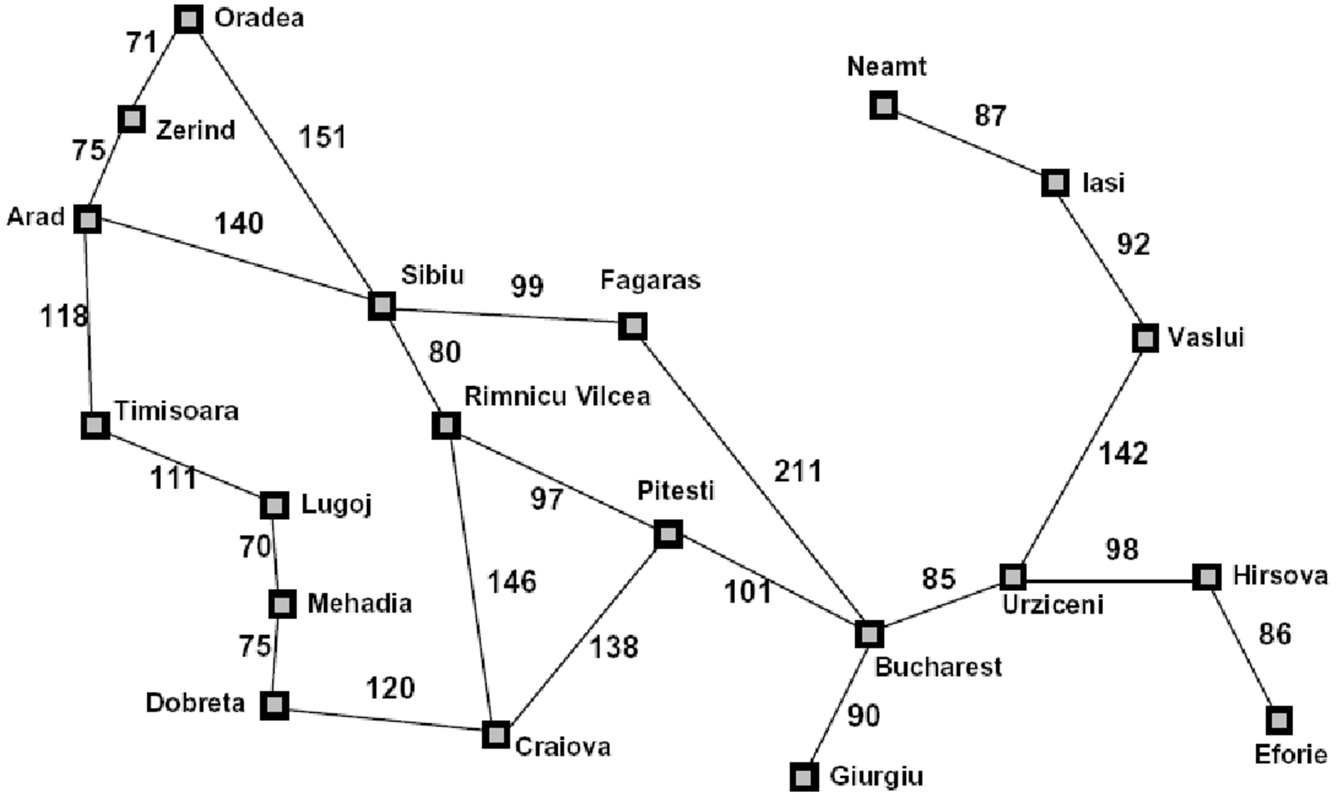
- A heuristic is:
  - A function that estimates how close a state is to a goal
  - Designed for a particular search problem
  - Pathing?
  - Examples: Manhattan distance, Euclidean distance for pathing







Example: Heuristic Function (Euclidean distance to Bucharest)



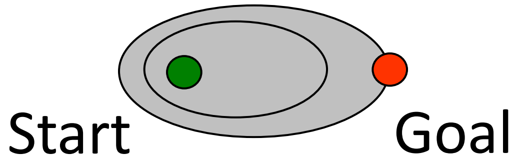
Straight−line distance to Bucharest

Arad	366
Bucharest	0
Craiova	160
Dobreta	242
Eforie	161
Fagaras	178
Giurgiu	77
Hirsova	151
Iasi	226
Lugoj	244
Mehadia	241
Neamt	234
Oradea	380
Pitesti	98
Rimnicu Vilcea	193
Sibiu	253
Timisoara	329
Urziceni	80
Vaslui	199
Zerind	374

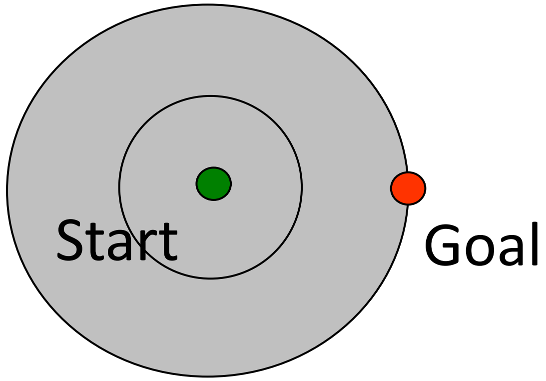
h(state) → value

Effect of heuristics

- Guide search ***towards the goal*** instead of ***all over the place***



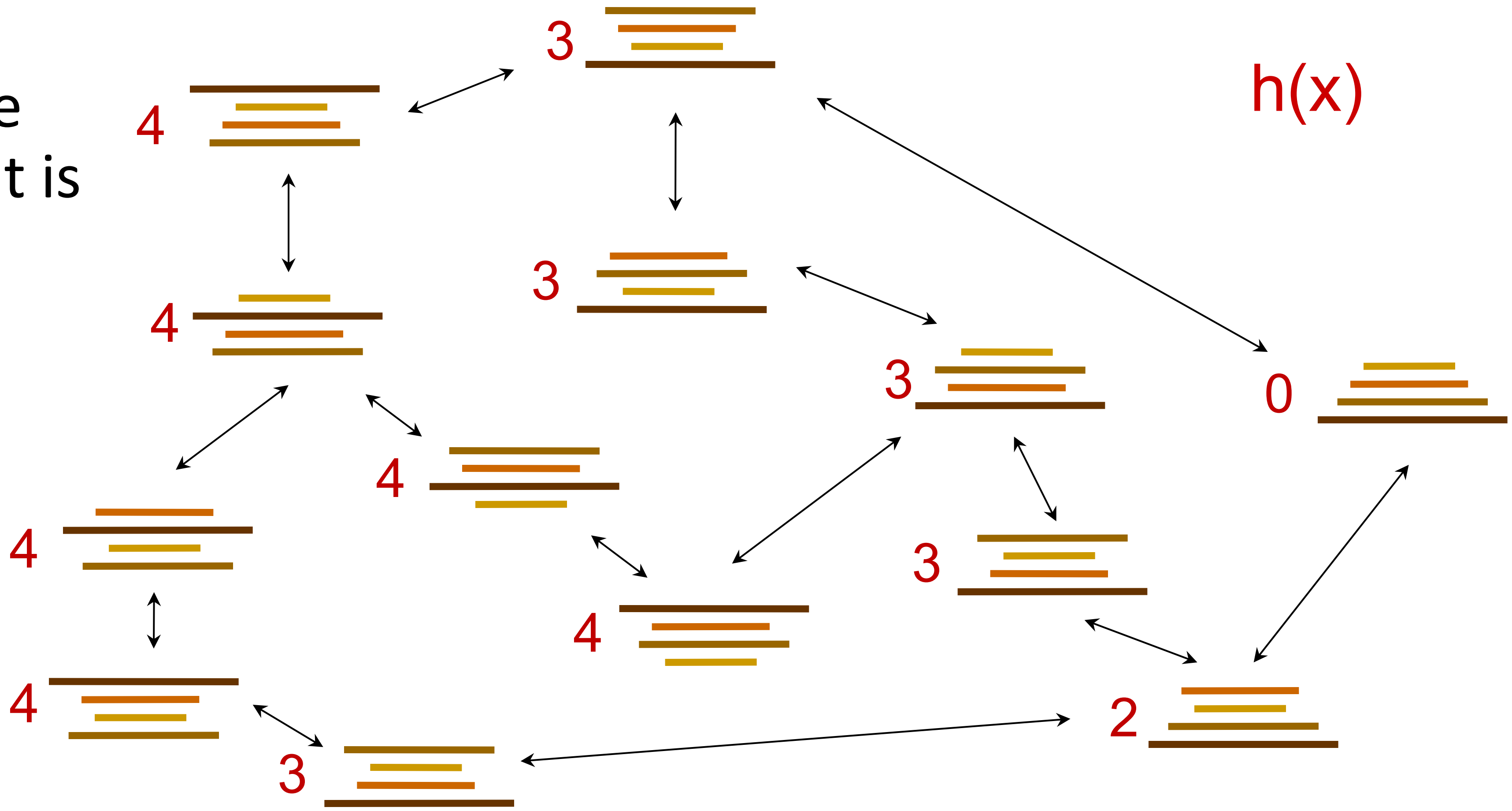
Informed



Uninformed

Example: Heuristic Function 2

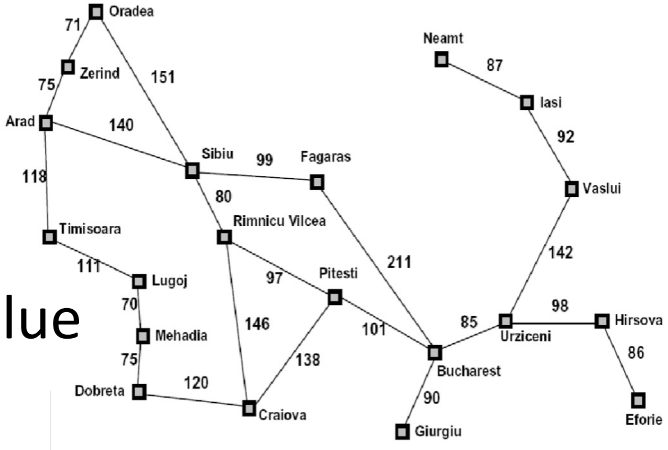
- Heuristic?
- E.g. the index of the largest pancake that is still out of place

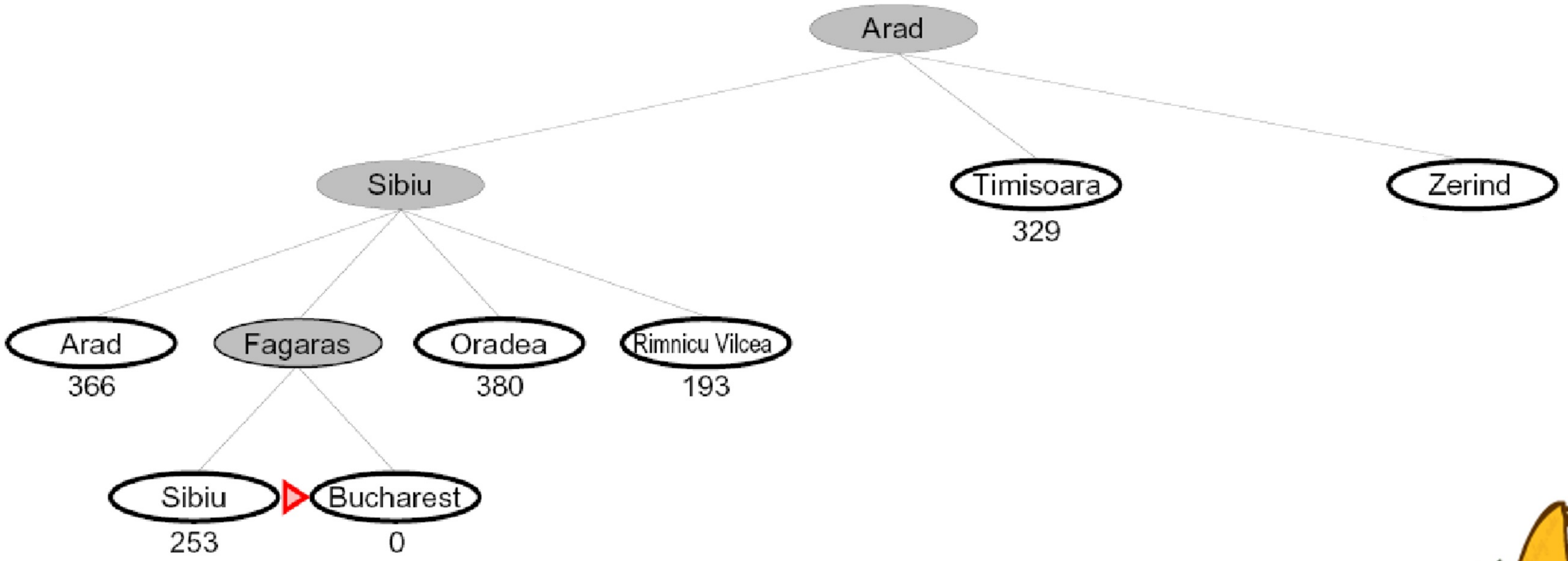


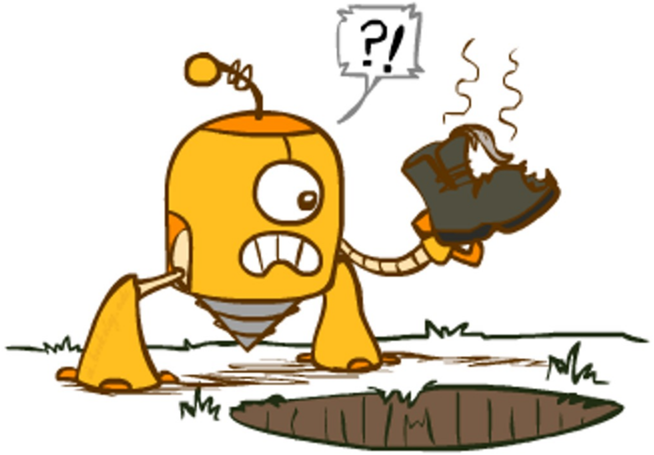
Greedy Search

Greedy Search

- Expand the node that seems closest to the goal, or least $h(n)$ value



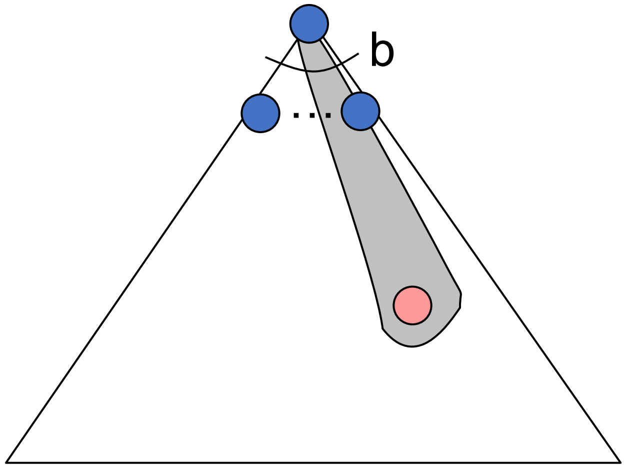




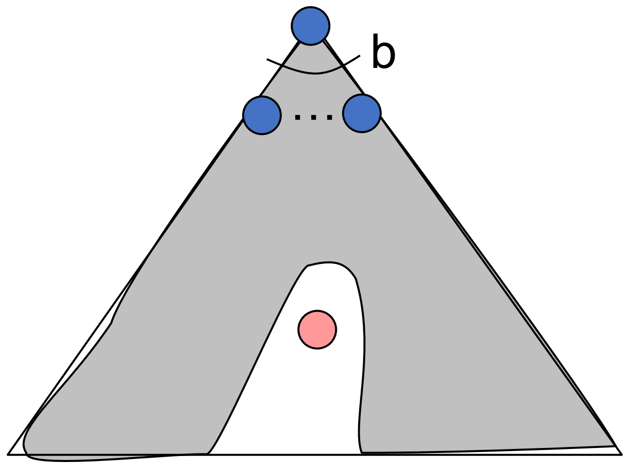
- Is it optimal?
  - No. Resulting path to Bucharest is not the shortest!
  - Why?
  - Heuristics might be wrong

Greedy Search 2

- Strategy: expand a node that you think is closest to a goal state
  - Heuristic: estimate of distance to nearest goal for each state



- A common case:
  - Best-first takes you straight to the (wrong) goal
  - (It chooses a node even if it's at the end of a very long and winding road)
- Worst-case: like a badly-guided DFS
  - (It takes h literally even if it's completely wrong)



[Demo: contours greedy empty (L3D1)]
[Demo: contours greedy pacman small maze (L3D4)]

Video of Demo Contours Greedy (Empty)

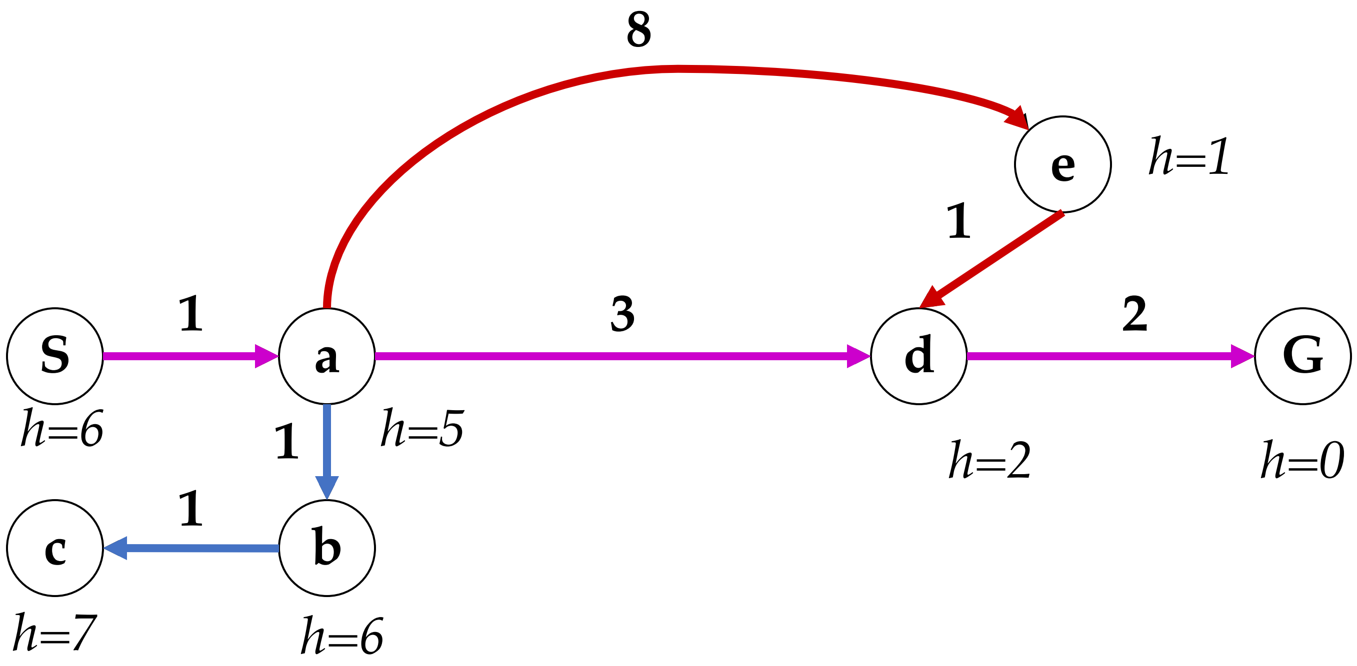
Video of Demo Contours Greedy (Pacman Small Maze)

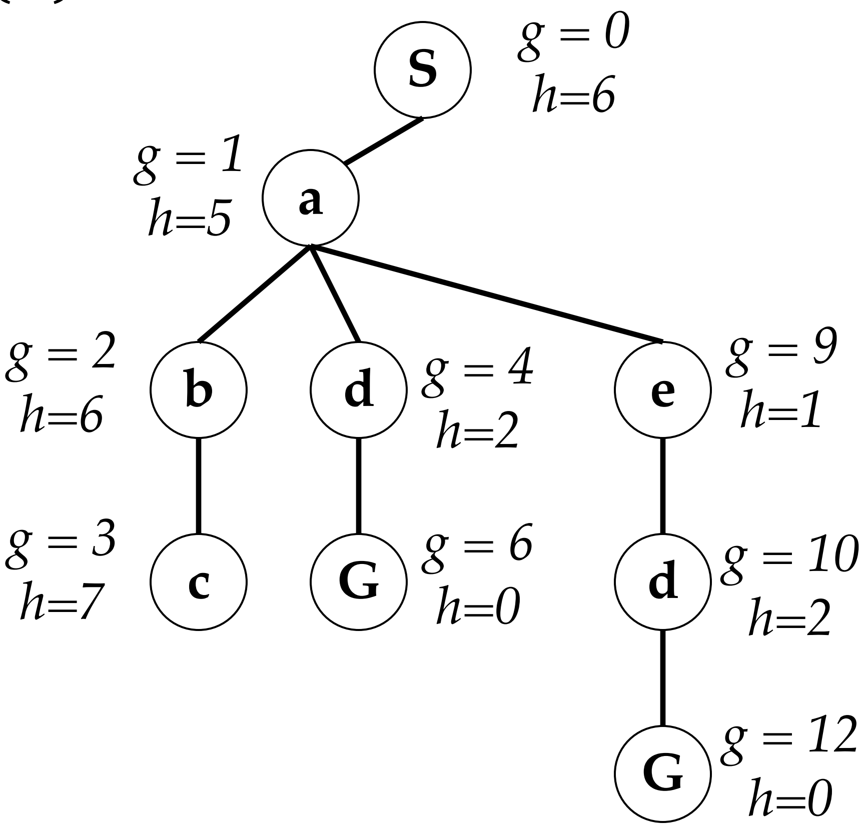
A* Search

A* Search

Combining UCS and Greedy

- Uniform-cost orders by path cost, or *backward cost* $g(n)$
- Greedy orders by goal proximity, or *forward cost* $h(n)$



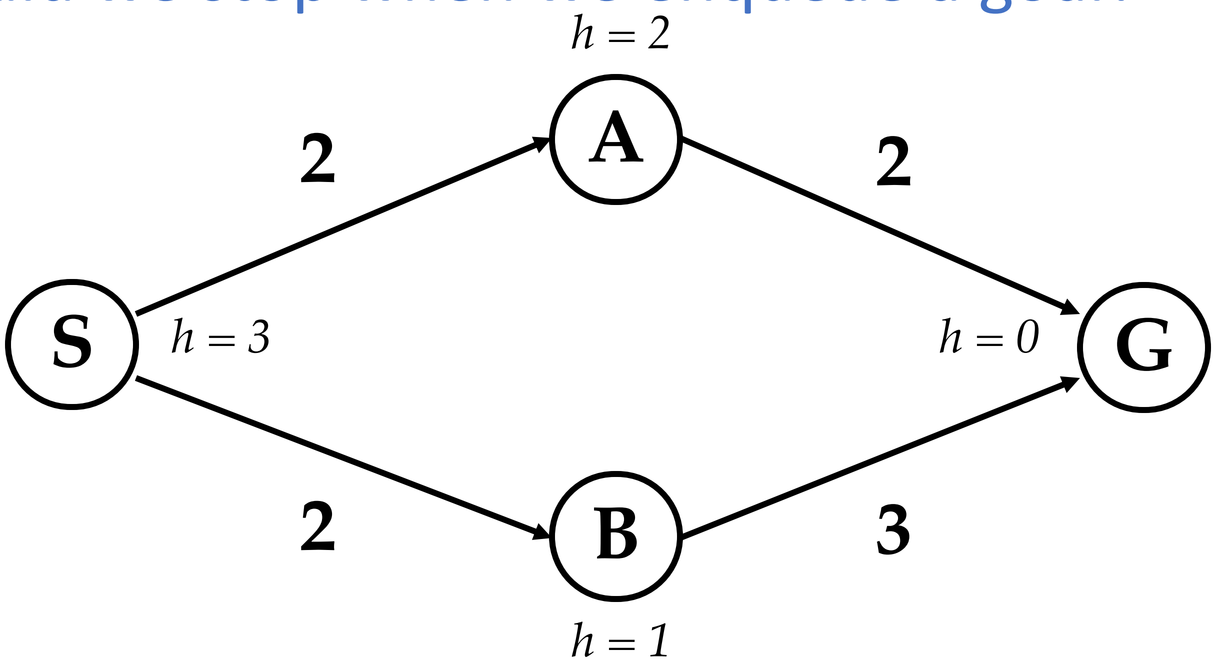


- A* Search orders by the sum: $f(n) = g(n) + h(n)$

Example: Teg Grenager

When should A* terminate?

- Should we stop when we enqueue a goal?

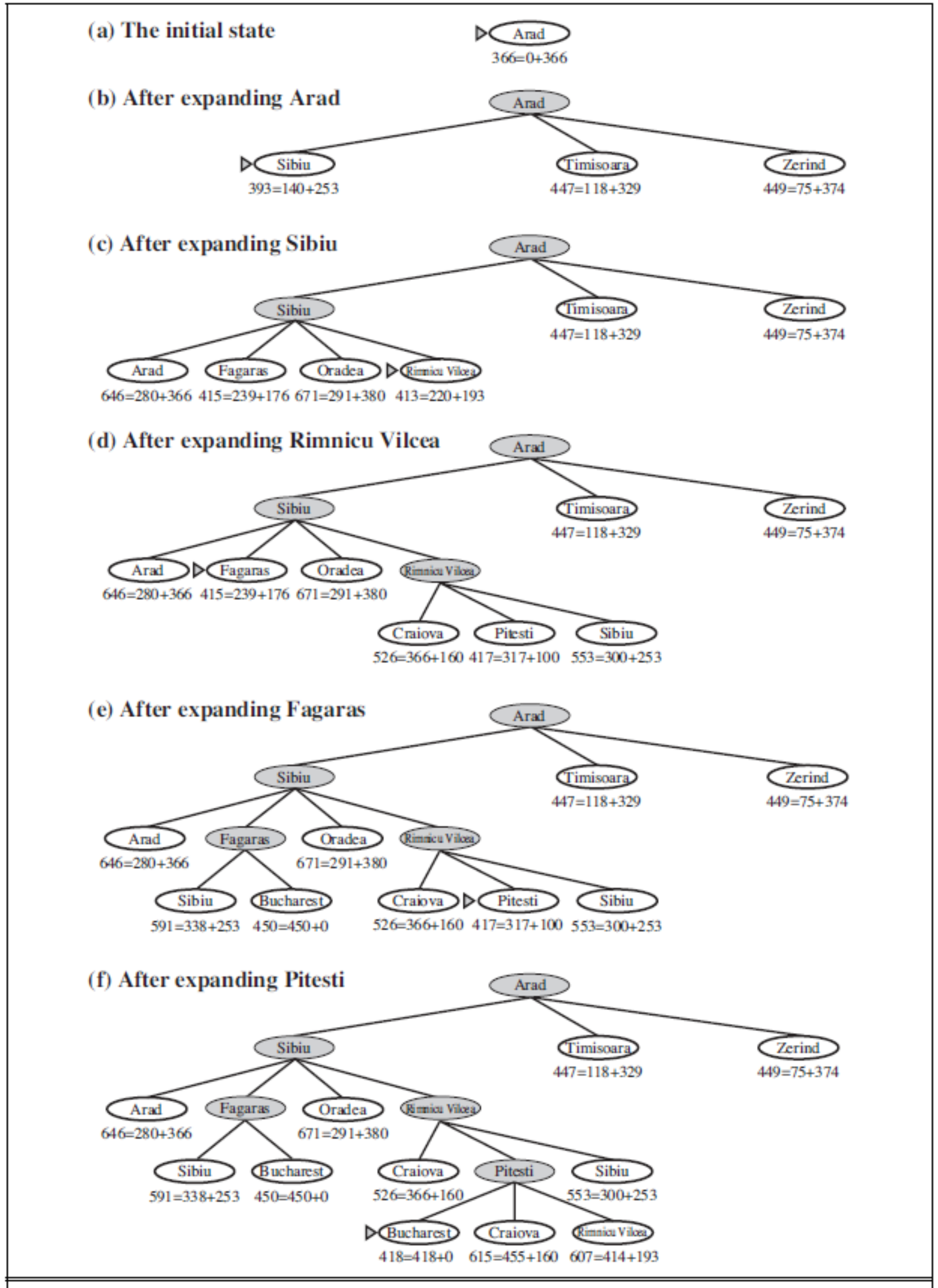


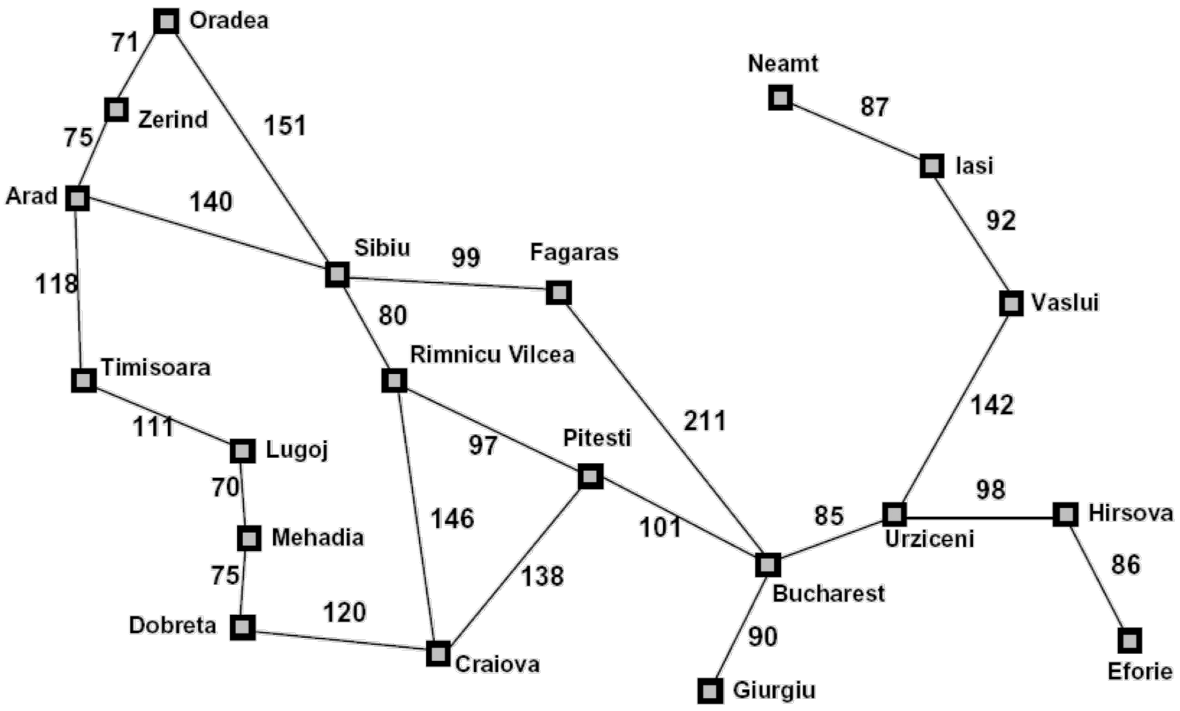
	g h +
~~S~~	~~0 3 3~~
~~S->A~~	~~2 2 4~~
~~S->B~~	~~2 1 3~~
S->B->G	5 0 5
S->A->G	**4 0 4**

- No: only stop when we dequeue a goal

A* Search

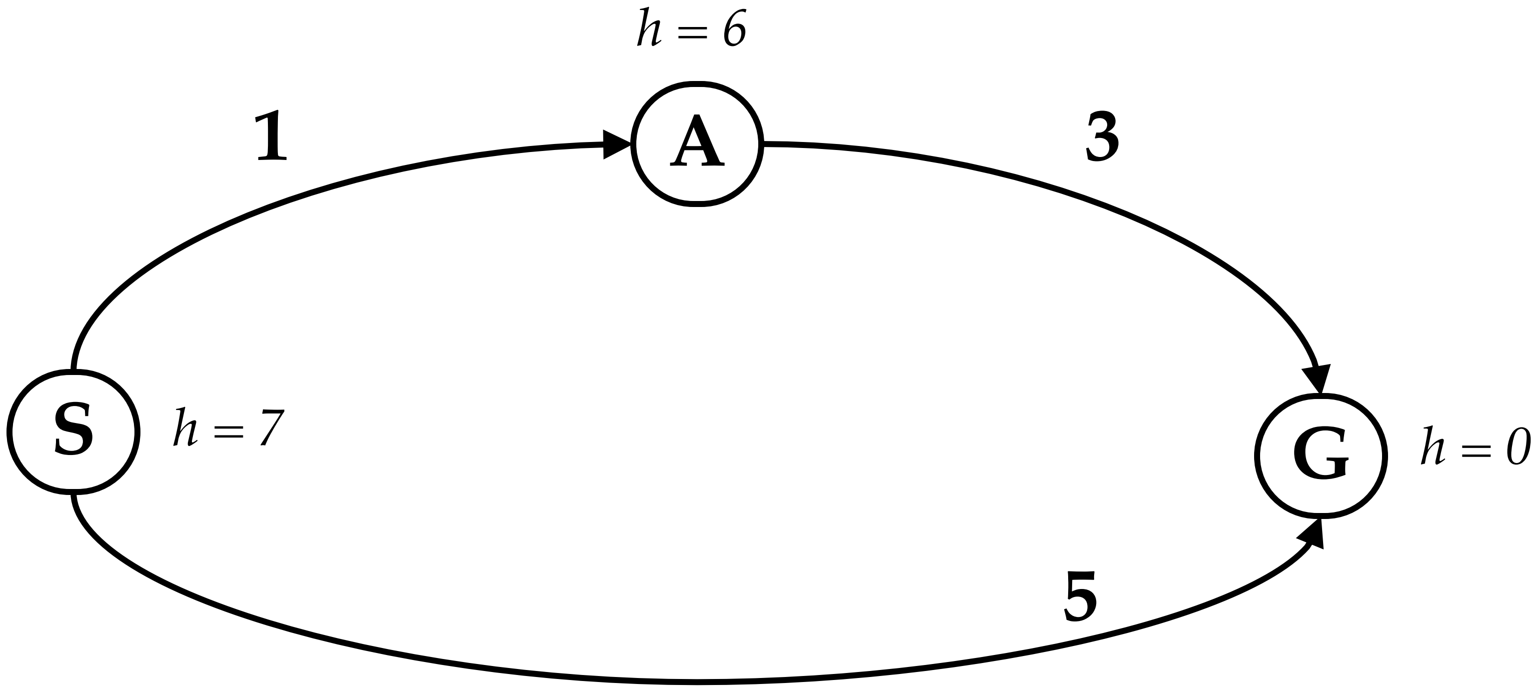
```
function A-STAR-SEARCH(problem) returns a solution, or failure
    initialize the frontier as a priority queue using f(n)=g(n)+h(n) as the priority
    add initial state of problem to frontier with priority f(S)=0+h(S)
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        for each resulting child from node
            add child to the frontier with f(n)=g(n)+h(n)
```





Arad	366	**Mehadia**	241
Bucharest	0	**Neamt**	234
Craiova	160	**Oradea**	380
Drobeta	242	**Pitesti**	100
Eforie	161	**Rimnicu Vilcea**	193
Fagaras	176	**Sibiu**	253
Giurgiu	77	**Timisoara**	329
Hirsova	151	**Urziceni**	80
Iasi	226	**Vaslui**	199
Lugoj	244	**Zerind**	374

Is A* Optimal?



		g	h	+
~~S~~		~~0~~	~~7~~	~~7~~
S->A		1	6	7
S->G		5	0	5

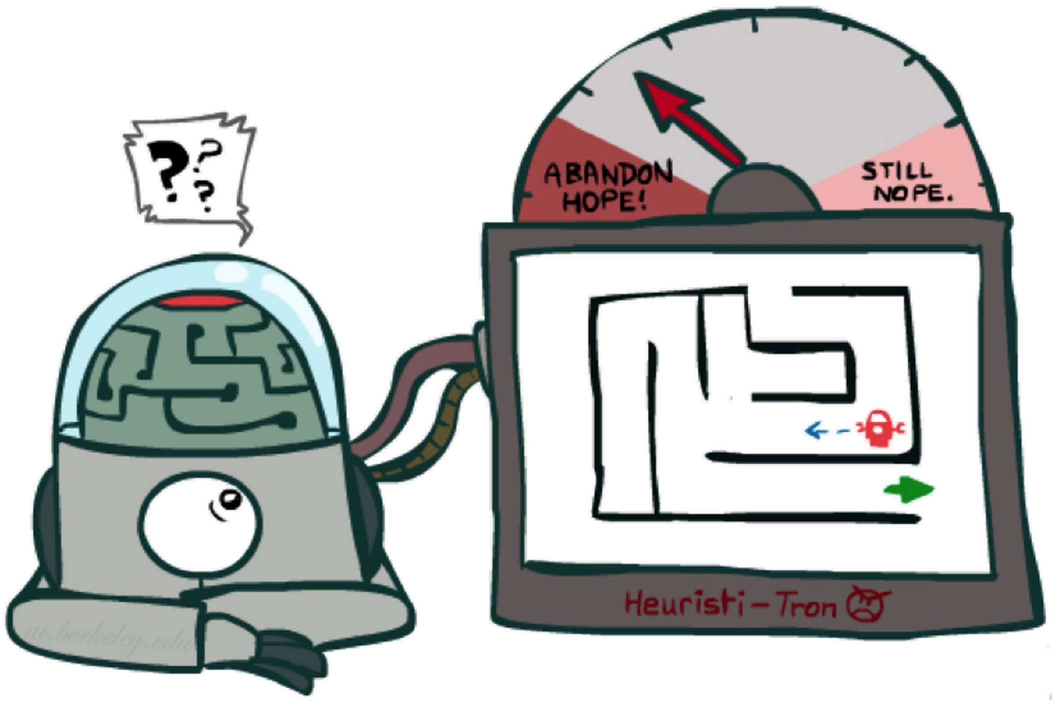
- What went wrong?
- Actual bad goal cost < estimated good goal cost
- We need estimates to be less than actual costs!

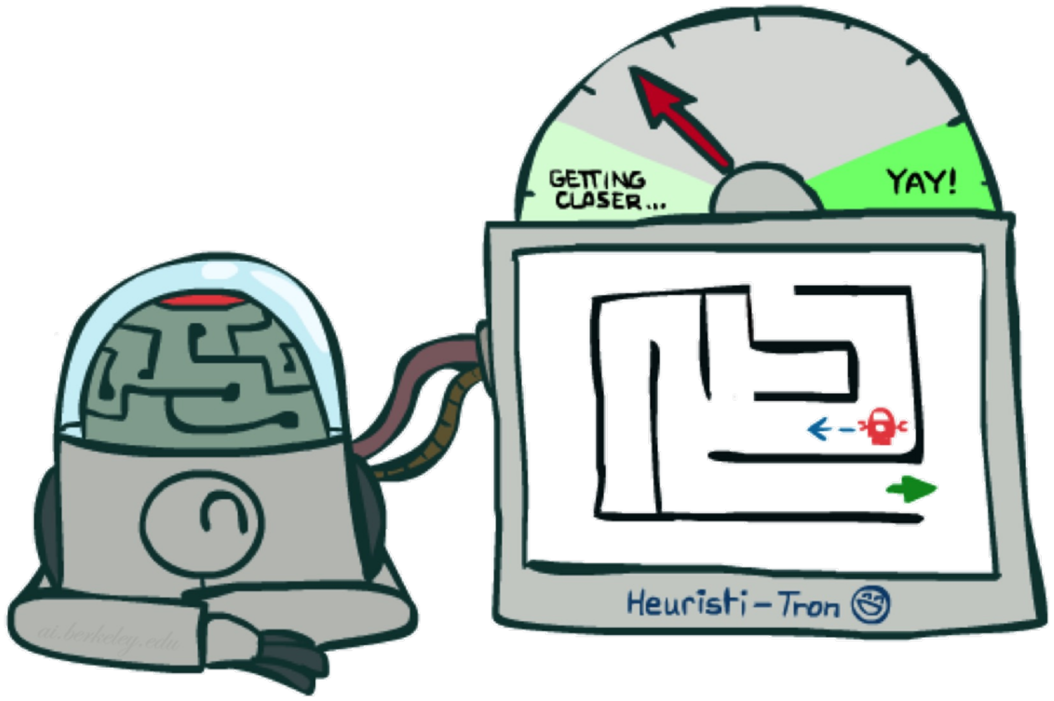
The Price is Wrong…

- Closest bid without going over…

https://www.youtube.com/watch?v=9B0ZKRurC5Y

Admissible Heuristics: Ideas





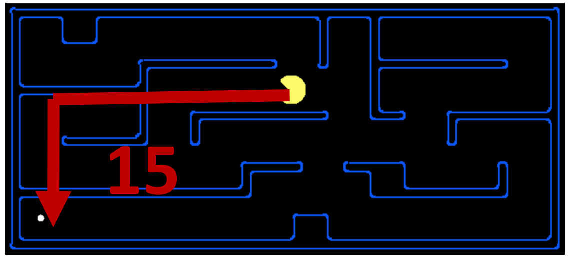
Inadmissible (pessimistic) heuristics break optimality by trapping good plans on the fringe

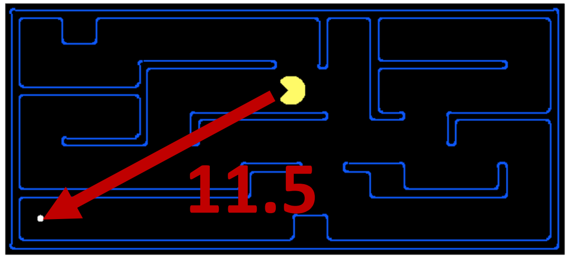
Admissible (optimistic) heuristics slow down bad plans but never outweigh true costs

Admissible Heuristics

- A heuristic h is *admissible* (optimistic) if
$$0 \le h(n) \le h^*(n)$$
where $h^*(n)$ is the true cost to a nearest goal

- Examples:





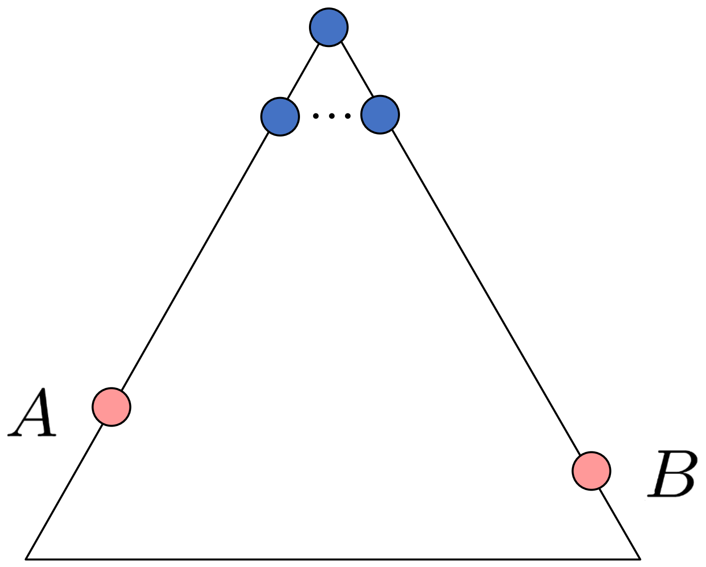
0.0

- Coming up with admissible heuristics is most of what’s involved in using A* in practice

Optimality of A* Tree Search

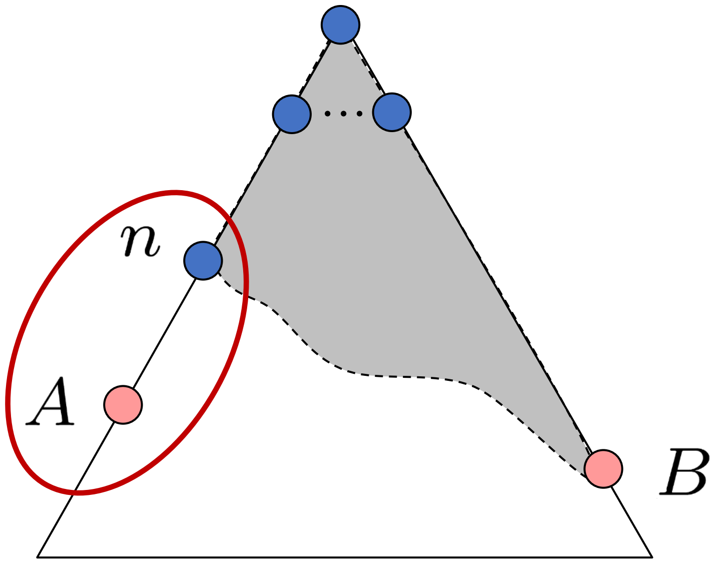
- Assume:
  - A is an optimal goal node
  - B is a suboptimal goal node
  - h is admissible

- Claim:
  - A will exit the fringe before B



Optimality of A* Tree Search: Blocking

- Proof:
  - Imagine B is on the fringe
  - Some ancestor n of A is on the fringe, too (maybe A!)
  - Claim: n will be expanded before B
    1. f(n) is less or equal to f(A)



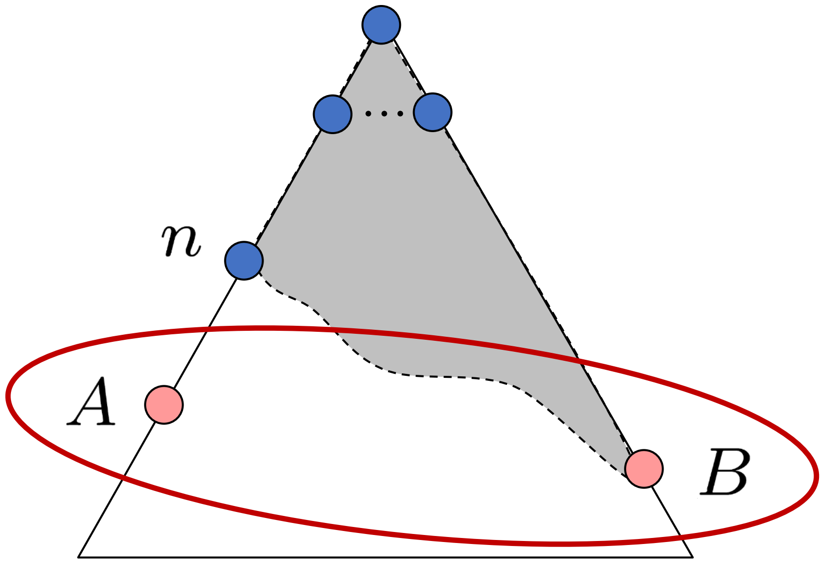
$$f(n) = g(n) + h(n)$$ Definition of f-cost

$$f(n) \leq g(A)$$ Admissibility of h

$$g(A) = f(A)$$ h = 0 at a goal

Optimality of A* Tree Search: Blocking 2

- Proof:
  - Imagine B is on the fringe
  - Some ancestor n of A is on the fringe, too (maybe A!)
  - Claim: n will be expanded before B
    1. f(n) is less or equal to f(A)
    2. f(A) is less than f(B)

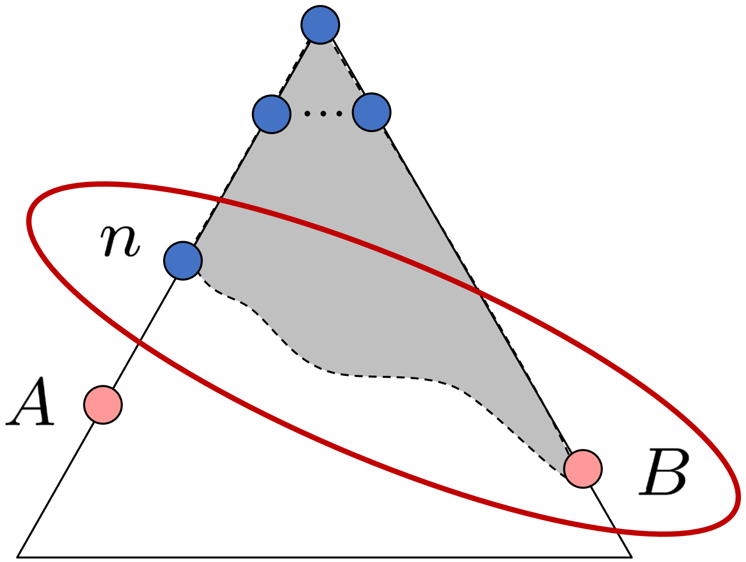


$$g(A) < g(B)$$ B is suboptimal

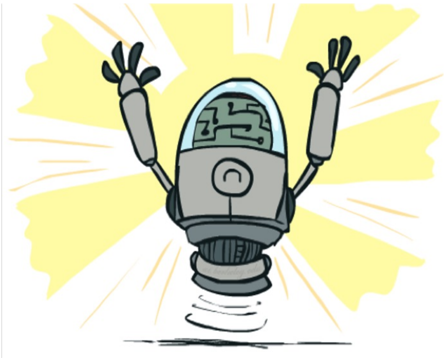
$$f(A) < f(B)$$ h = 0 at a goal

Optimality of A* Tree Search: Blocking 3

- Proof:
  - Imagine B is on the fringe
  - Some ancestor n of A is on the fringe, too (maybe A!)
  - Claim: n will be expanded before B
    1. f(n) is less or equal to f(A)
    2. f(A) is less than f(B)
    3. *n* expands before B
  - All ancestors of A expand before B
  - A expands before B
  - A* search is optimal

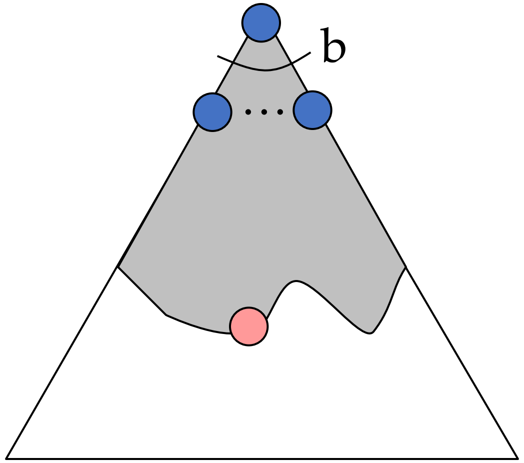


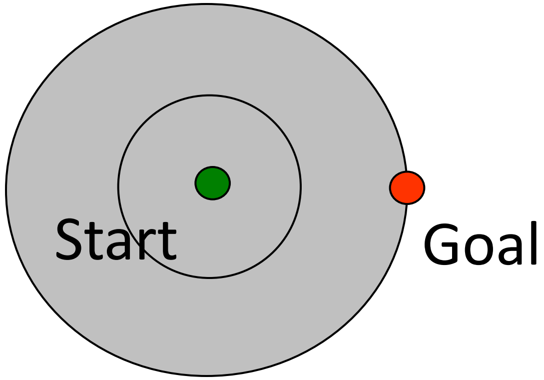
$$f(n) \leq f(A) < f(B)$$



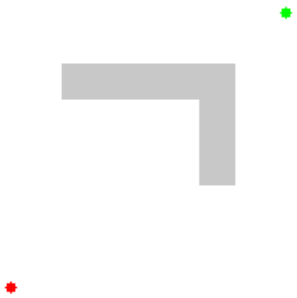
UCS vs A*

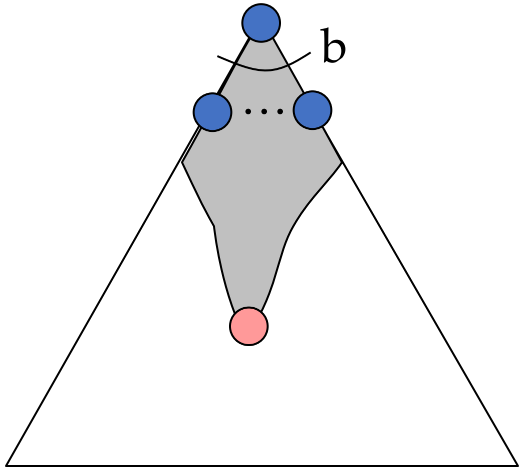
- Uniform-cost expands equally in all “directions”

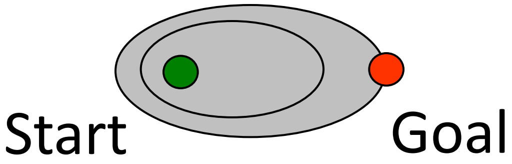




- A* expands mainly toward the goal, but does hedge its bets to ensure optimality







[Demo: contours UCS / greedy / A* empty (L3D1)]
[Demo: contours A* pacman small maze (L3D5)]

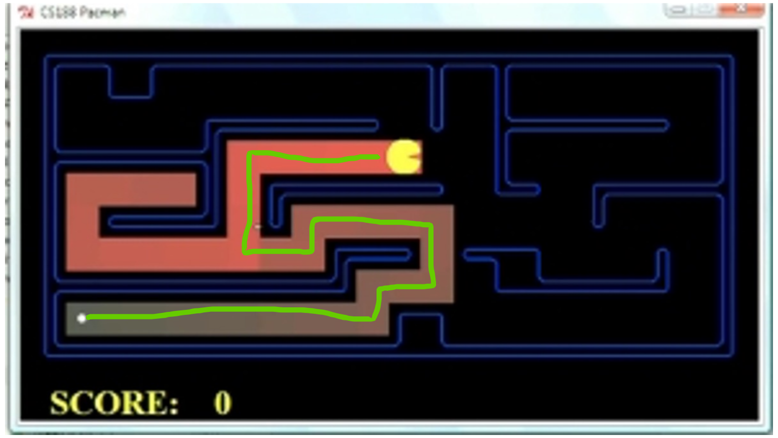
Video of Demo Contours (Empty) -- UCS

Video of Demo Contours (Empty) -- Greedy

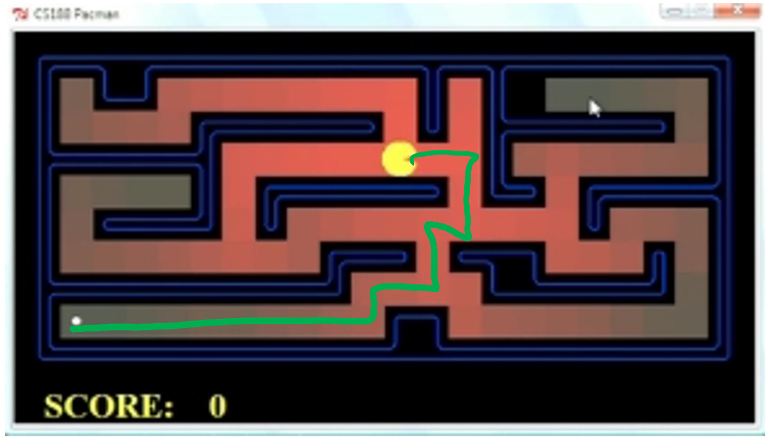
Video of Demo Contours (Empty) – A*

Video of Demo Contours (Pacman Small Maze) – A*

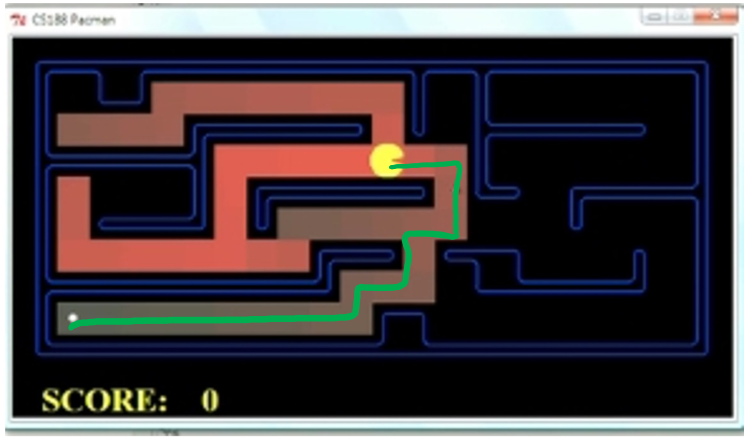
Comparison



Greedy



Uniform Cost



A*

A* Applications

- Video games
- Pathing / routing problems
- Resource planning problems
- Robot motion planning
- Language analysis
- Machine translation
- Speech recognition
- …

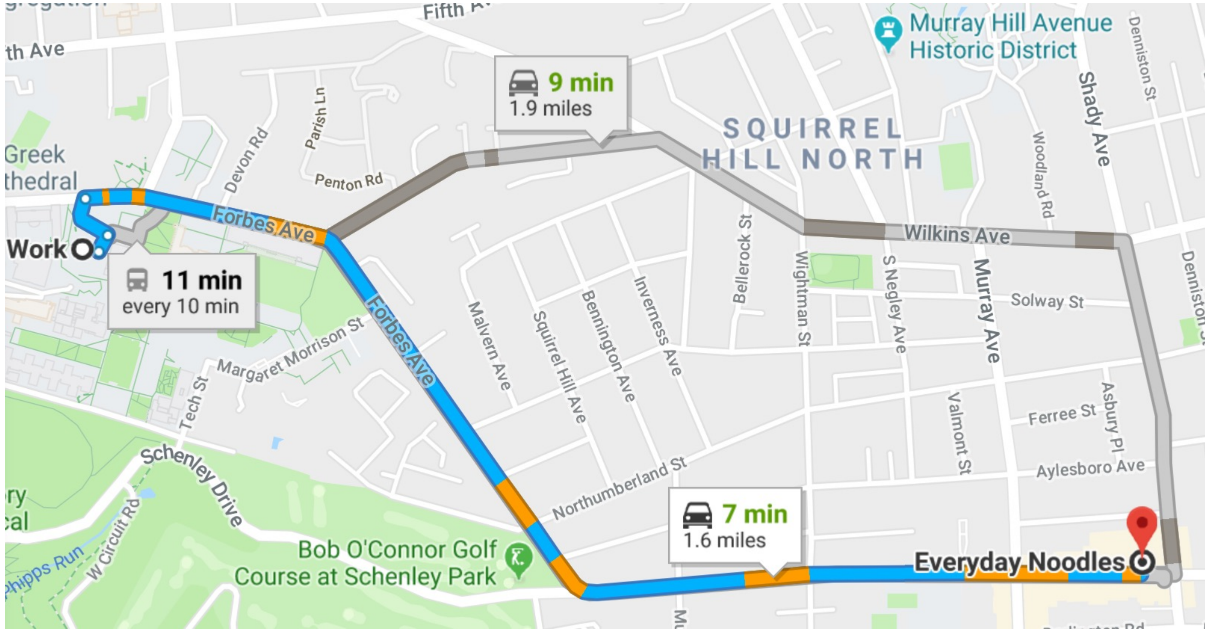
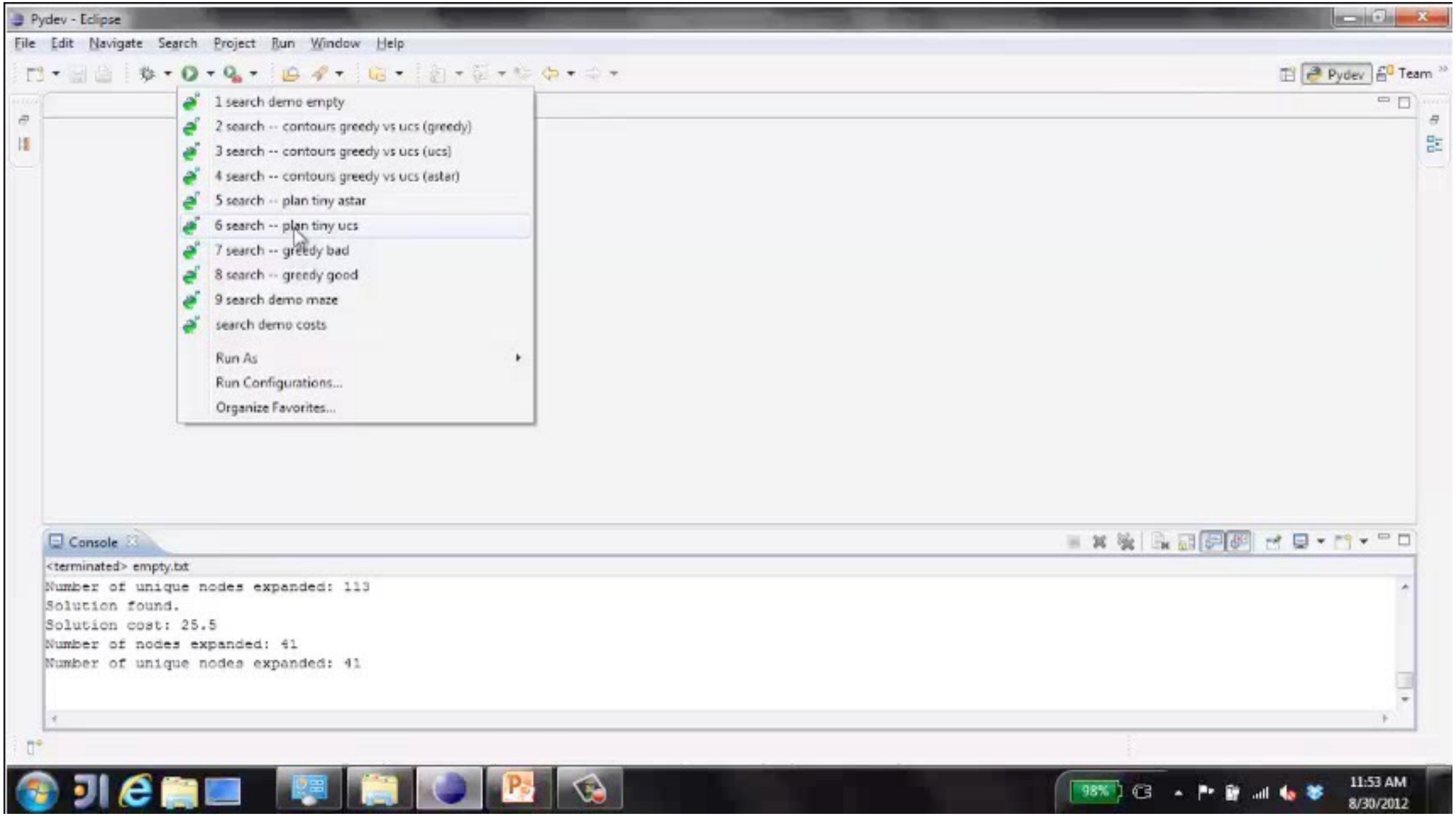


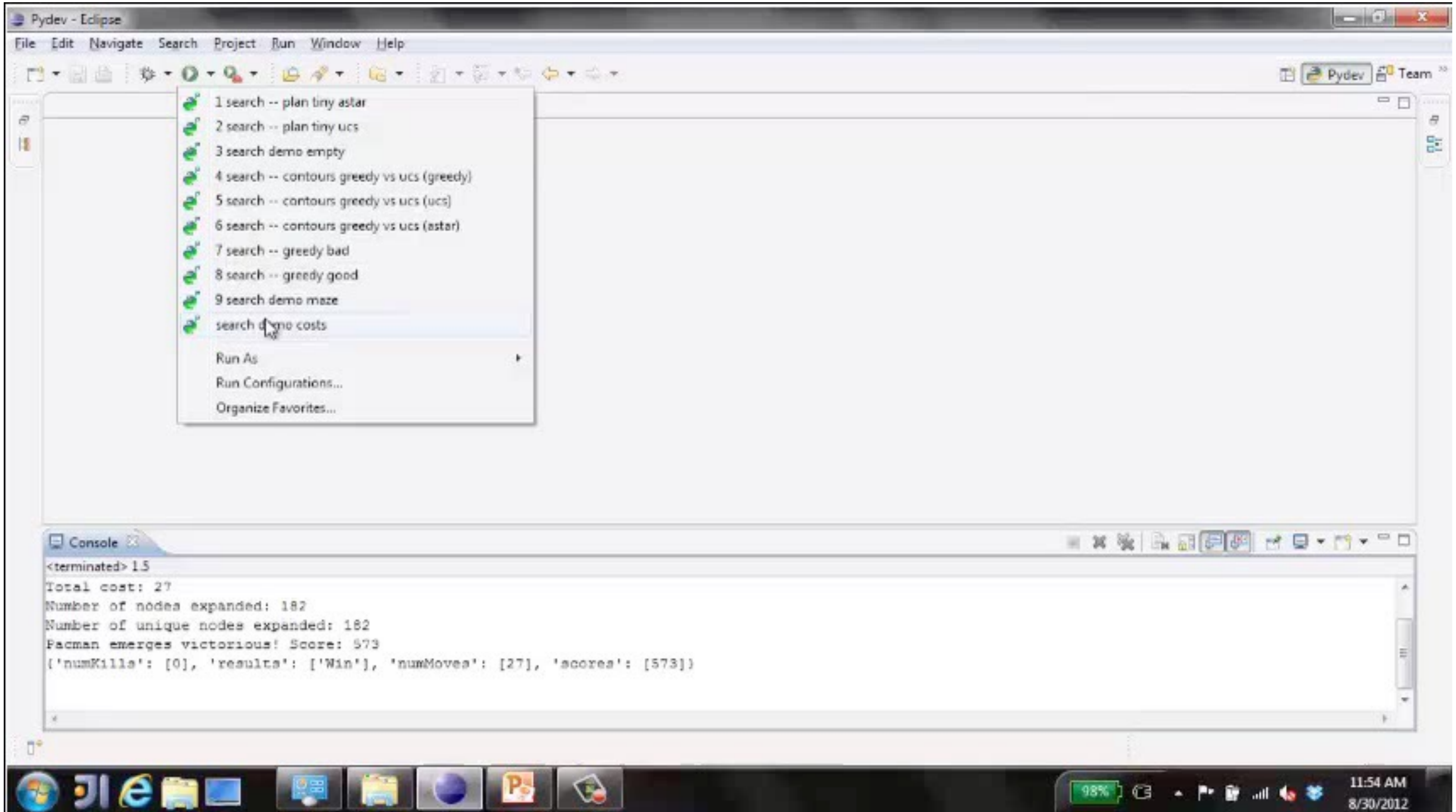
Image: maps.google.com

[Demo: UCS / A* pacman tiny maze (L3D6,L3D7)]
[Demo: guess algorithm Empty Shallow/Deep (L3D8)]

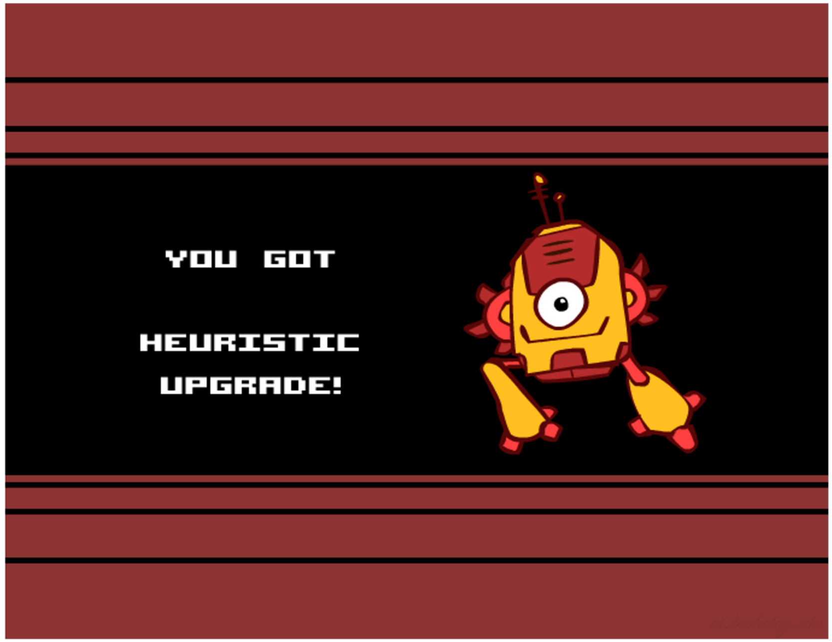
Video of Demo Pacman (Tiny Maze) – UCS / A*



Video of Demo Empty Water Shallow/Deep – Guess Algorithm

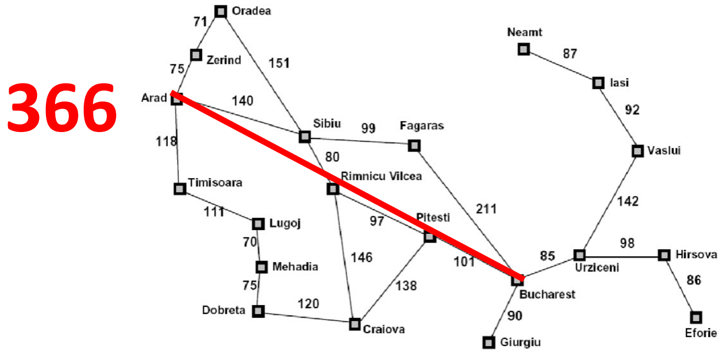


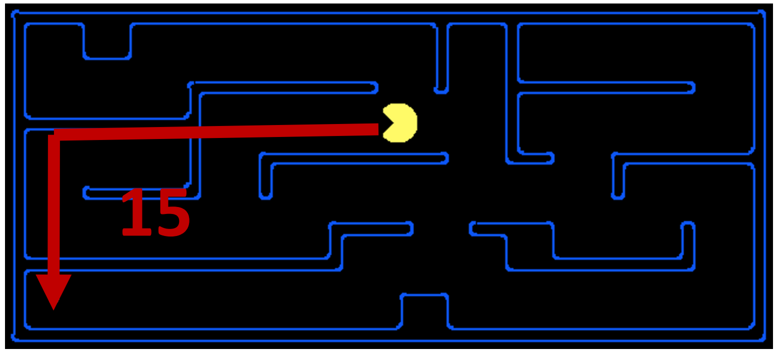
Creating Heuristics



- Most of the work in solving hard search problems optimally is in coming up with admissible heuristics

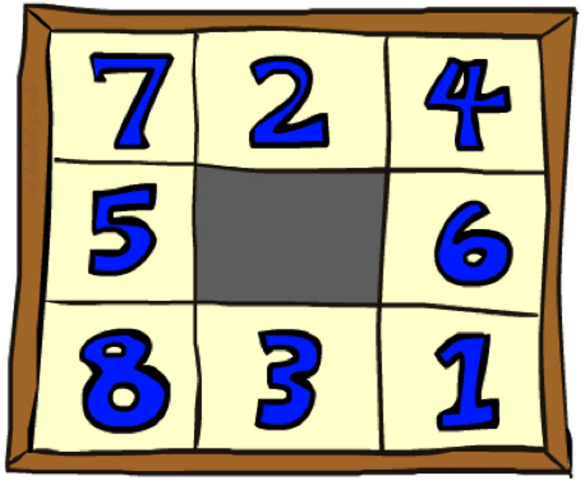
- Often, admissible heuristics are solutions to relaxed problems, where new actions are available



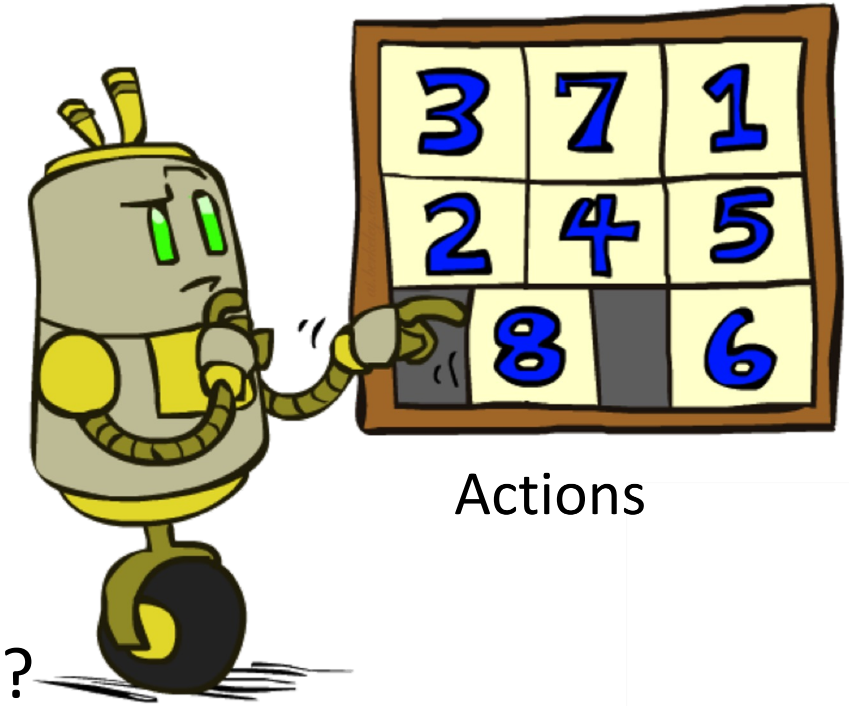


- Inadmissible heuristics are often useful too

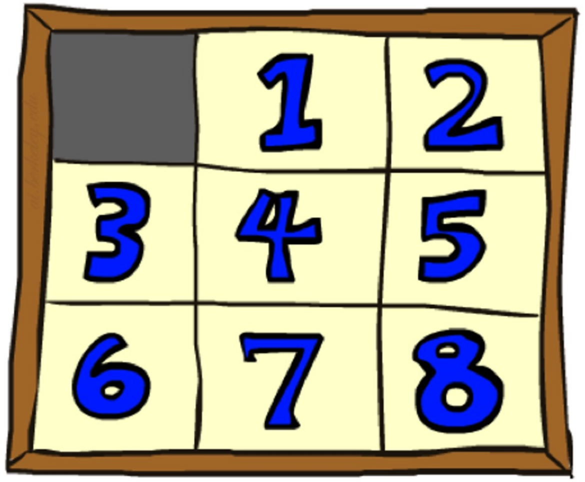
Example: 8 Puzzle



Start State



Actions



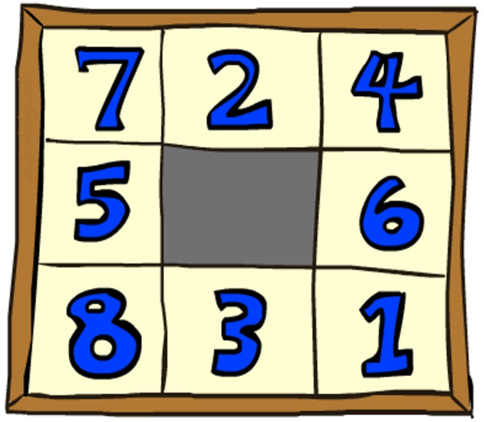
Goal State

- What are the states?
- How many states?
- What are the actions?
- How many successors from the start state?
- What should the costs be?

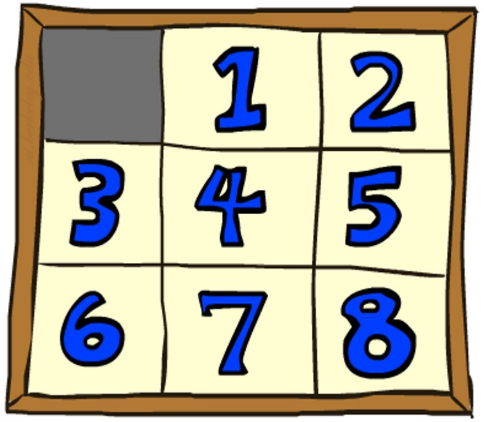
Admissible heuristics?

Example: 8 Puzzle - 2

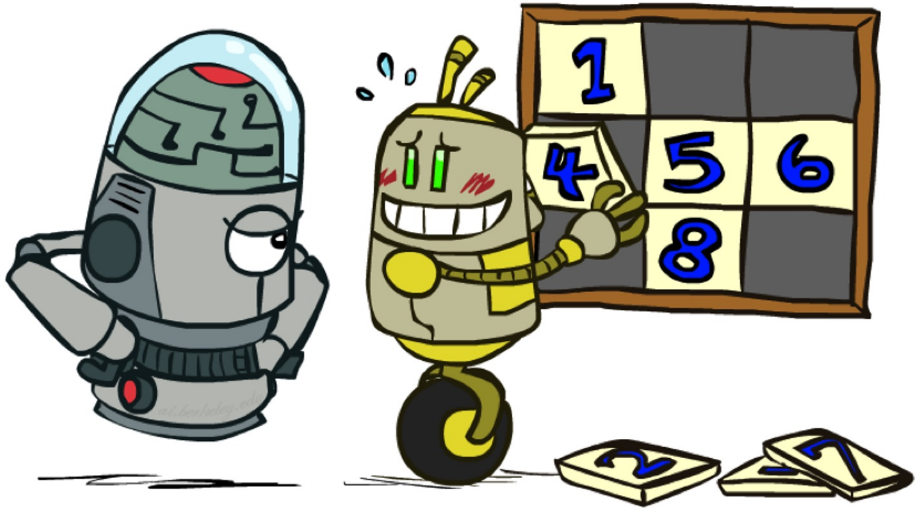
- Heuristic: Number of tiles misplaced
- Why is it admissible?
- h(start) = 8
- This is a relaxed-problem heuristic



Start State



Goal State

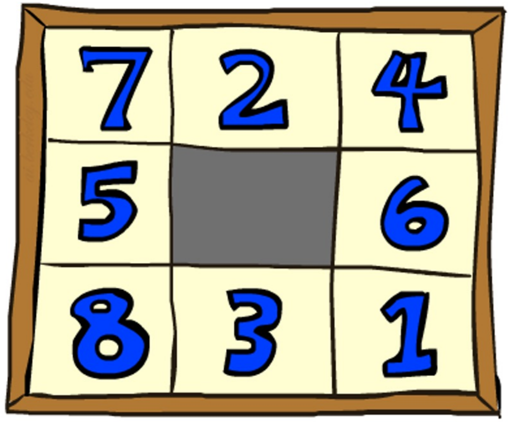


	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
UCS	112	6,300	3.6×10^6
TILES	13	39	227

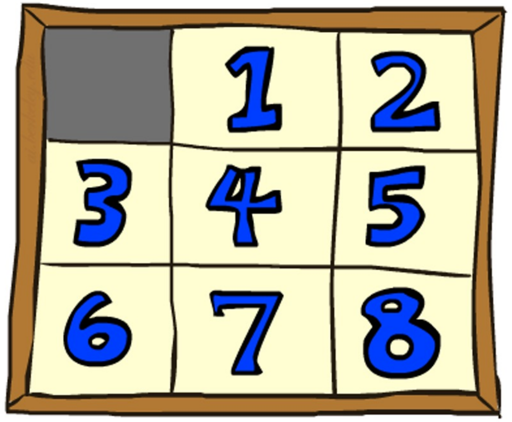
Statistics from Andrew Moore

Example: 8 Puzzle - 3

- What if we had an easier 8-puzzle where any tile could slide any direction at any time, ignoring other tiles?



Start State



Goal State

- Total Manhattan distance

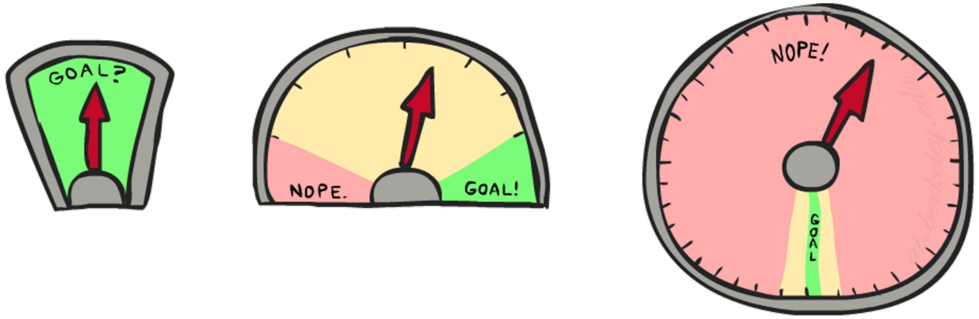
- Why is it admissible?

- h(start) = 3 + 1 + 2 + … = 18

	Average nodes expanded when the optimal path has…		
	…4 steps	…8 steps	…12 steps
TILES	13	39	227
MANHATTAN	12	25	73

Example: 8 Puzzle - 4

- How about using the actual cost as a heuristic?
  - Would it be admissible?
  - Would we save on nodes expanded?
  - What's wrong with it?



- With A*: a trade-off between quality of estimate and work per node
  - As heuristics get closer to the true cost, you will expand fewer nodes but usually do more work per node to compute the heuristic itself

Combining Heuristics, Dominance

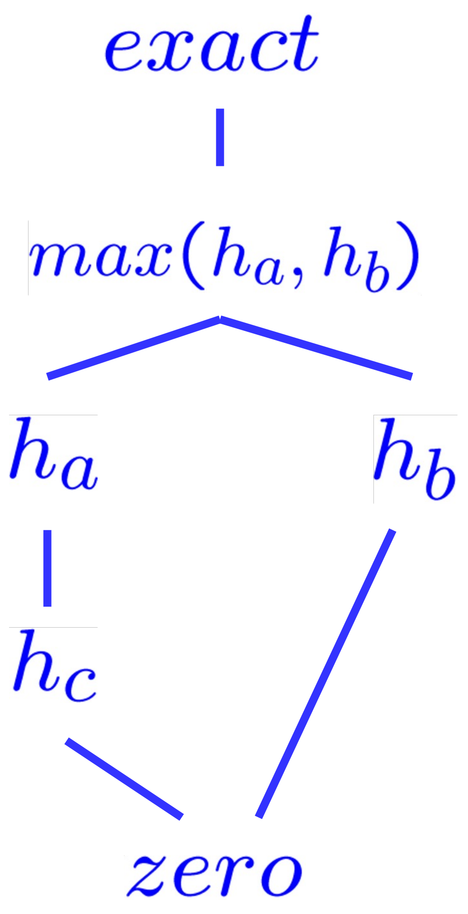
- Dominance: $h_a \geq h_c$ if

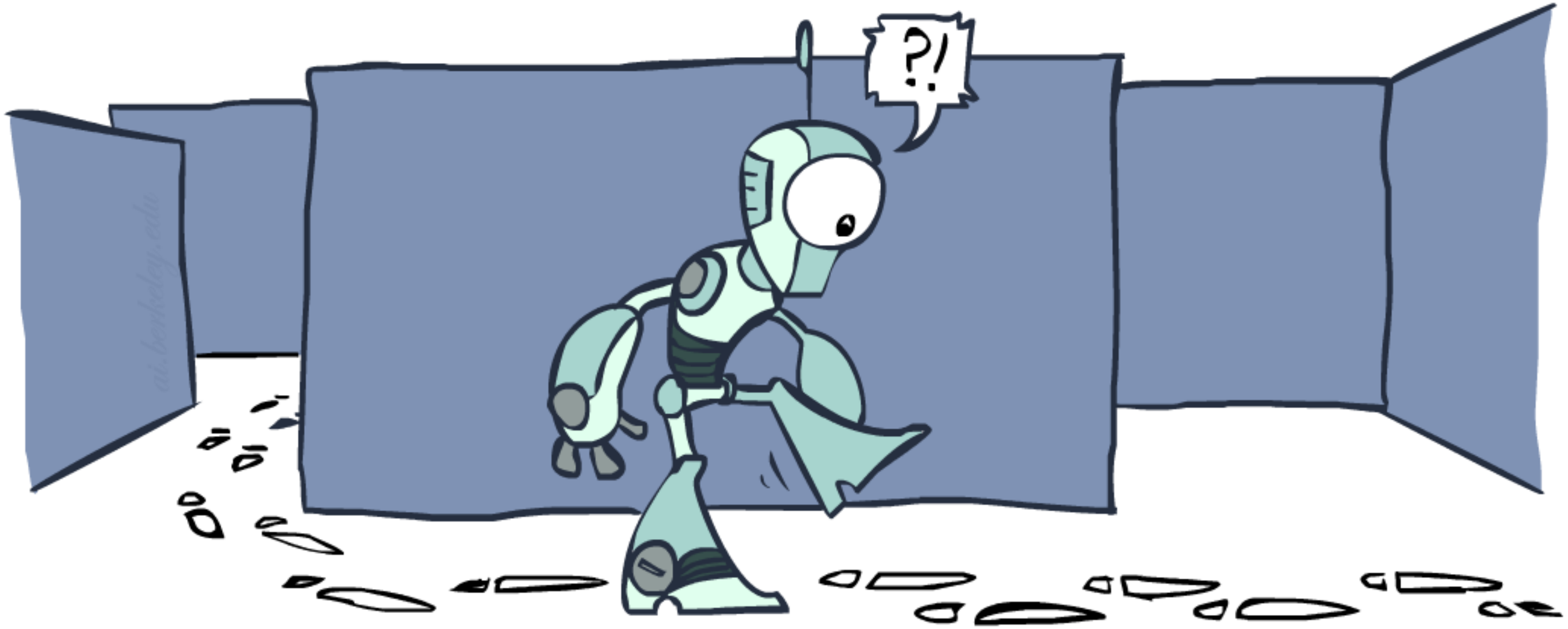
$$\forall n : h_a(n) \geq h_c(n)$$

  - Roughly speaking, larger is better as long as both are admissible
- Heuristics form a semi-lattice:
  - Max of admissible heuristics is admissible

$$h(n) = max(h_a(n), h_b(n))$$

- Trivial heuristics
  - Bottom of lattice is the zero heuristic (what does this give us?)
  - Top of lattice is the exact heuristic, but usually too expensive



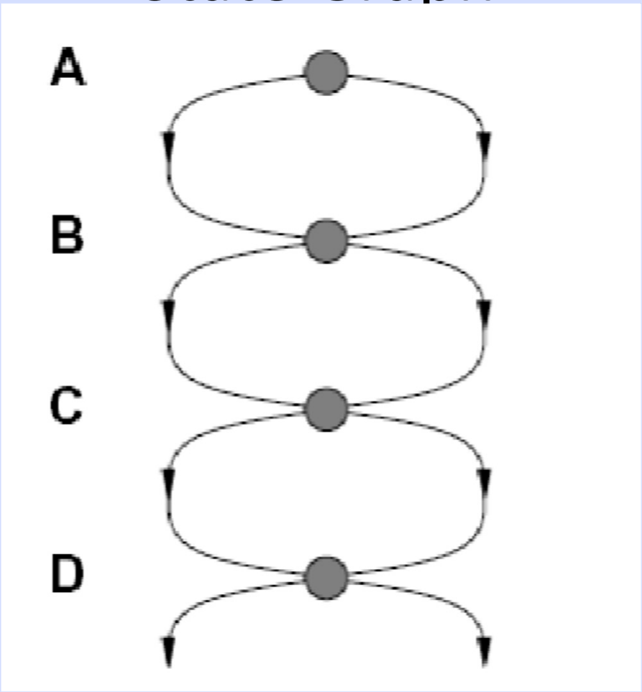


Graph Search

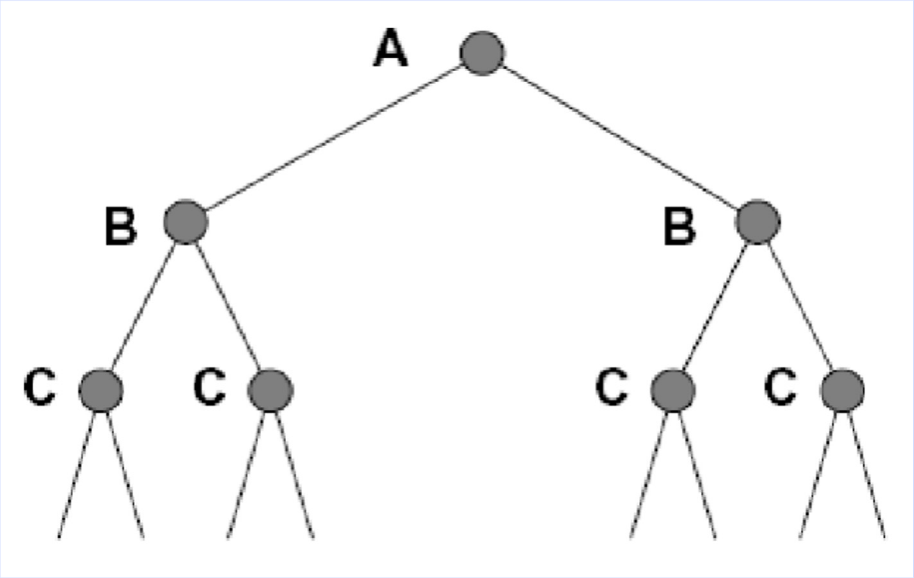
Tree Search: Extra Work!

- Failure to detect repeated states can cause exponentially more work

State Graph

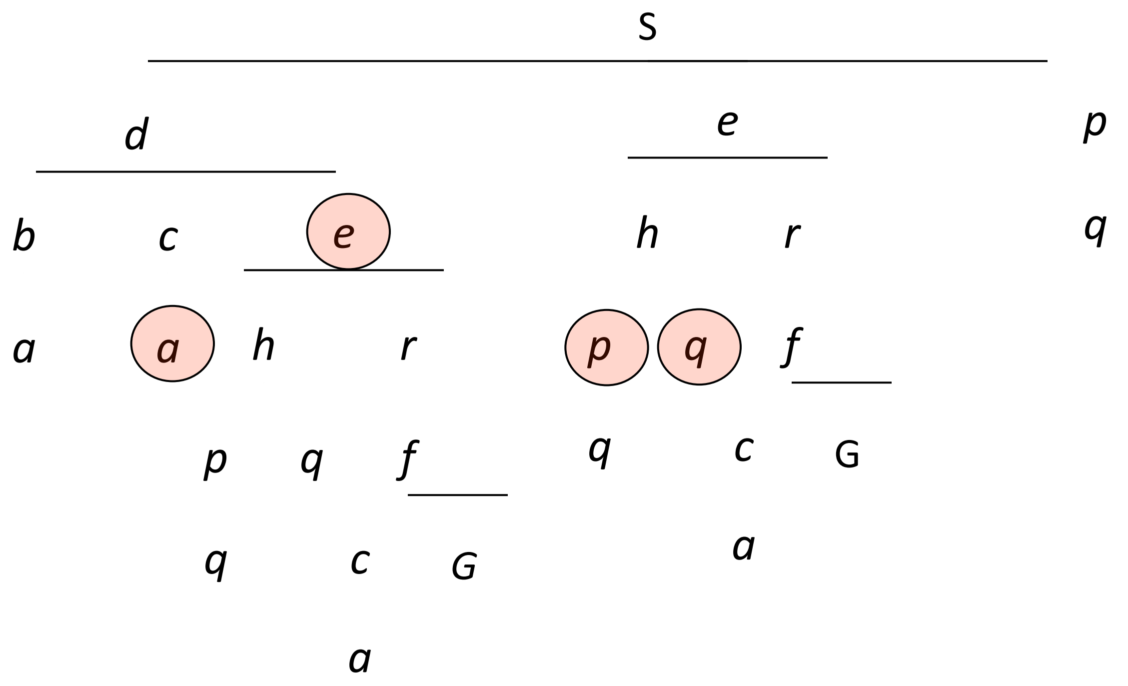


Search Tree



Graph Search

- In BFS, for example, we shouldn't bother expanding the circled nodes (why?)



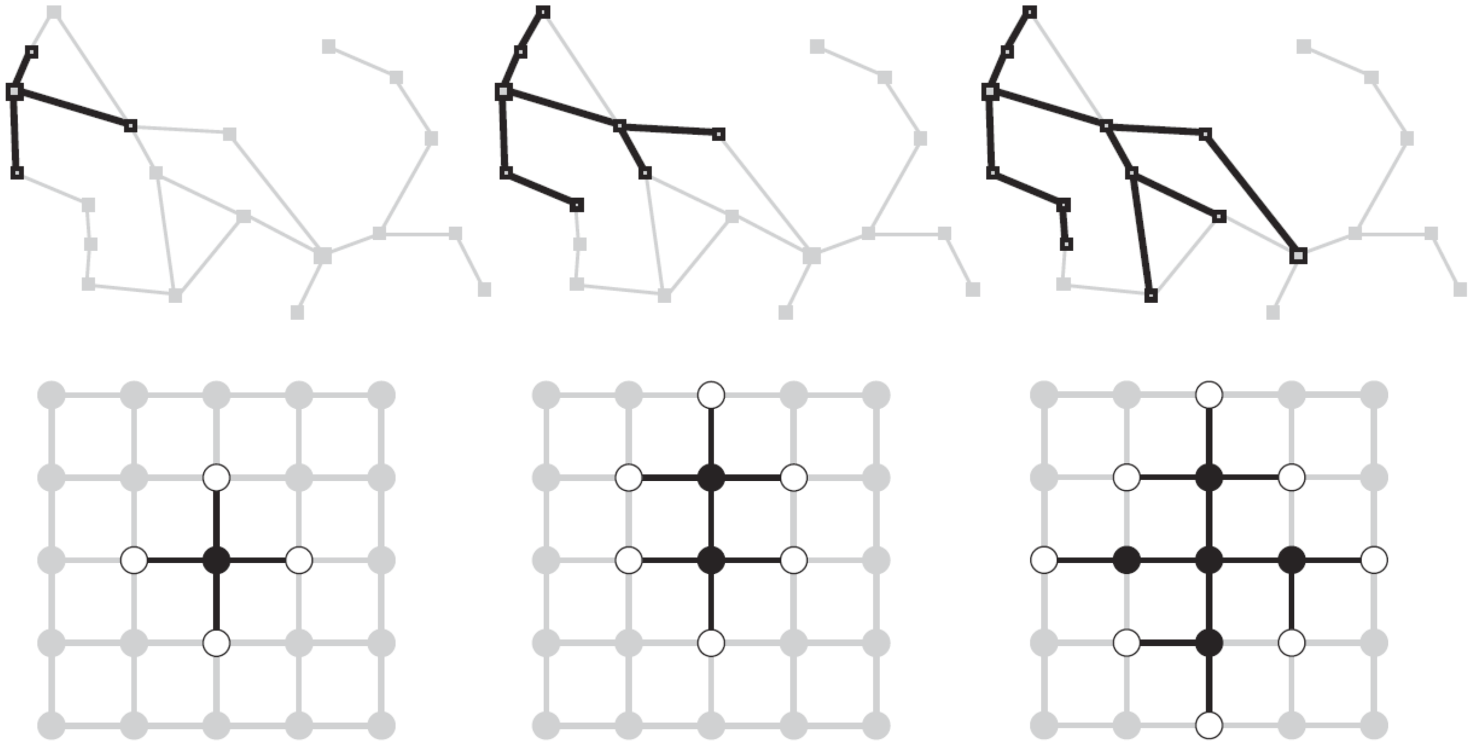
Graph Search 2

- Idea: never expand a state twice
- How to implement:
  - Tree search + set of expanded states (“closed set”, “explored set”)
  - Expand the search tree node-by-node, but…
  - Before expanding a node, check to make sure its state has never been expanded before
  - If not new, skip it, if new add to closed/explored set
- Important: store the closed/explored set as a set, not a list
- Can graph search wreck completeness? Why/why not?
- How about optimality?

```
function GRAPH_SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a specific work list (stack, queue, priority queue)
    add initial state of problem to frontier
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
```

Graph Search 3

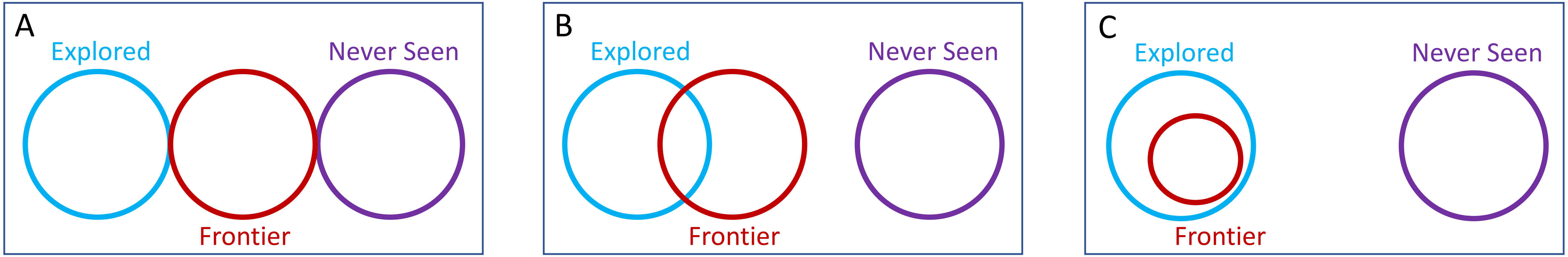
- This graph search algorithm overlays a tree on a graph
- The frontier states separate the explored states from never seen states



Images: AIMA, Figure 3.8, 3.9

Quiz

- What is the relationship between these sets of states after each loop iteration in GRAPH_SEARCH?
- (Loop invariants!!!)

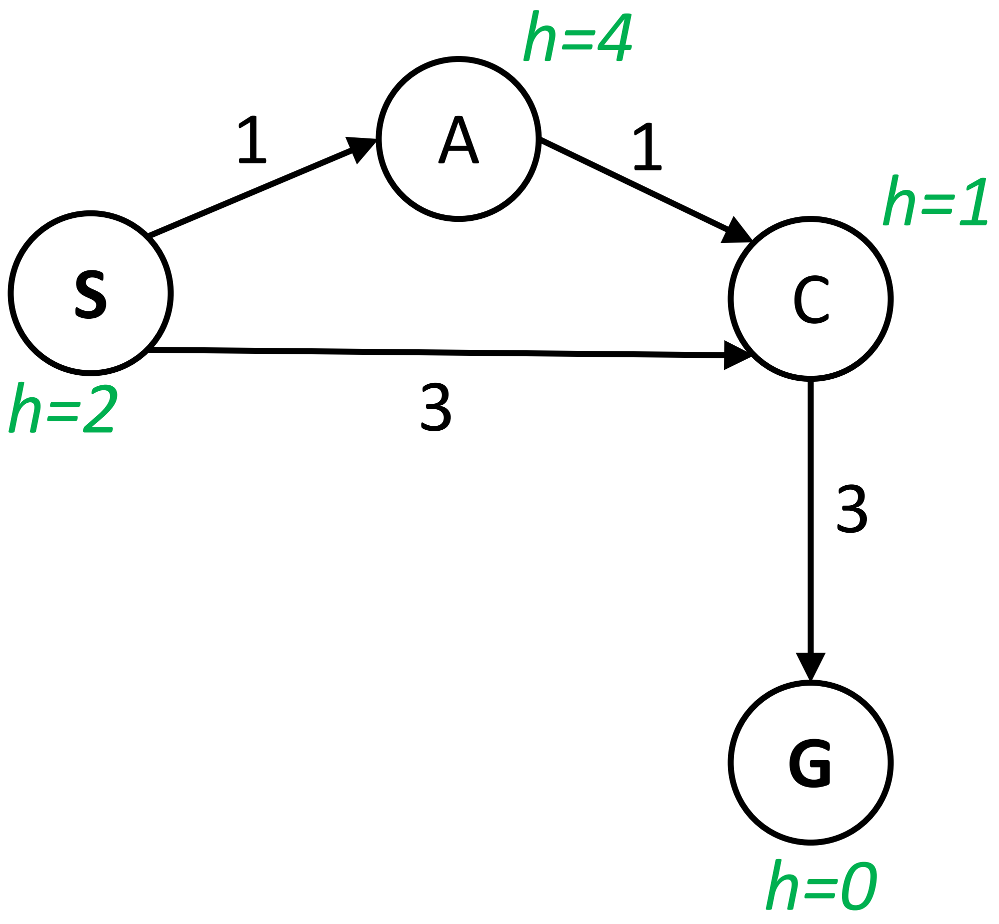


```
function UNIFORM-COST-GRAPH-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using node's path_cost as the priority
    add initial state of problem to frontier with path_cost = 0
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher path_cost then
                replace that frontier node with child
```

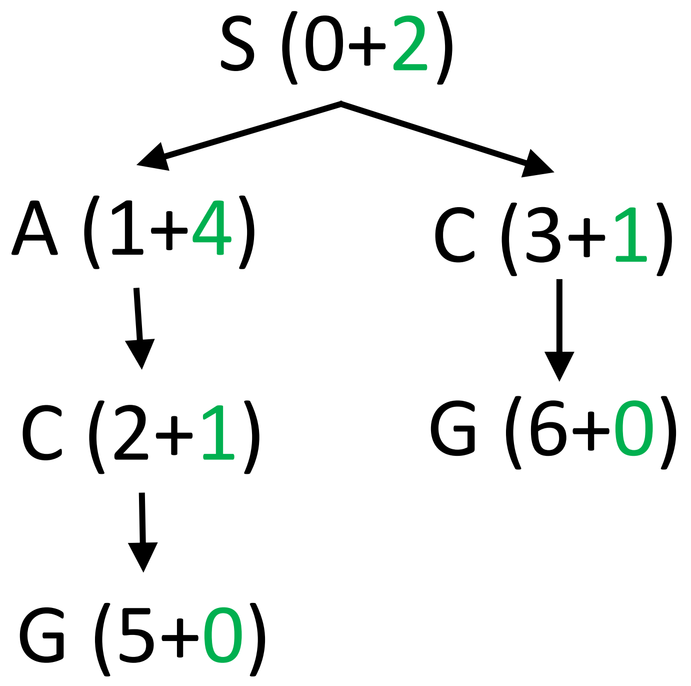
```
function A-STAR-GRAPH-SEARCH(problem) returns a solution, or failure
    initialize the explored set to be empty
    initialize the frontier as a priority queue using f(n) = g(n) + h(n) as the priority
    add initial state of problem to frontier with priority f(S) = 0 + h(S)
    loop do
        if the frontier is empty then
            return failure
        choose a node and remove it from the frontier
        if the node contains a goal state then
            return the corresponding solution
        add the node state to the explored set
        for each resulting child from node
            if the child state is not already in the frontier or explored set then
                add child to the frontier
            else if the child is already in the frontier with higher f(n) then
                replace that frontier node with child
```

A* Tree Search

State space graph

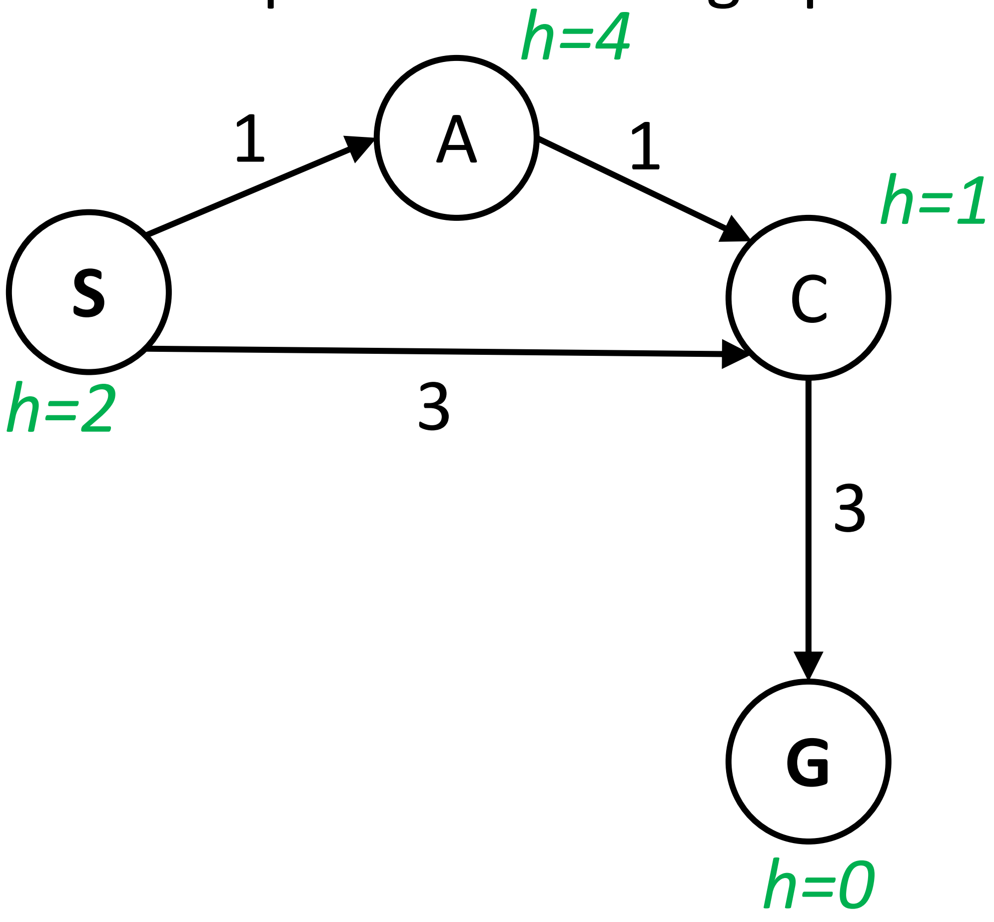


Search tree



Quiz: A* Graph Search

- What paths does A* graph search consider during its search?



A) ~~S~~, ~~S-A~~, ~~S-C~~, S-C-G

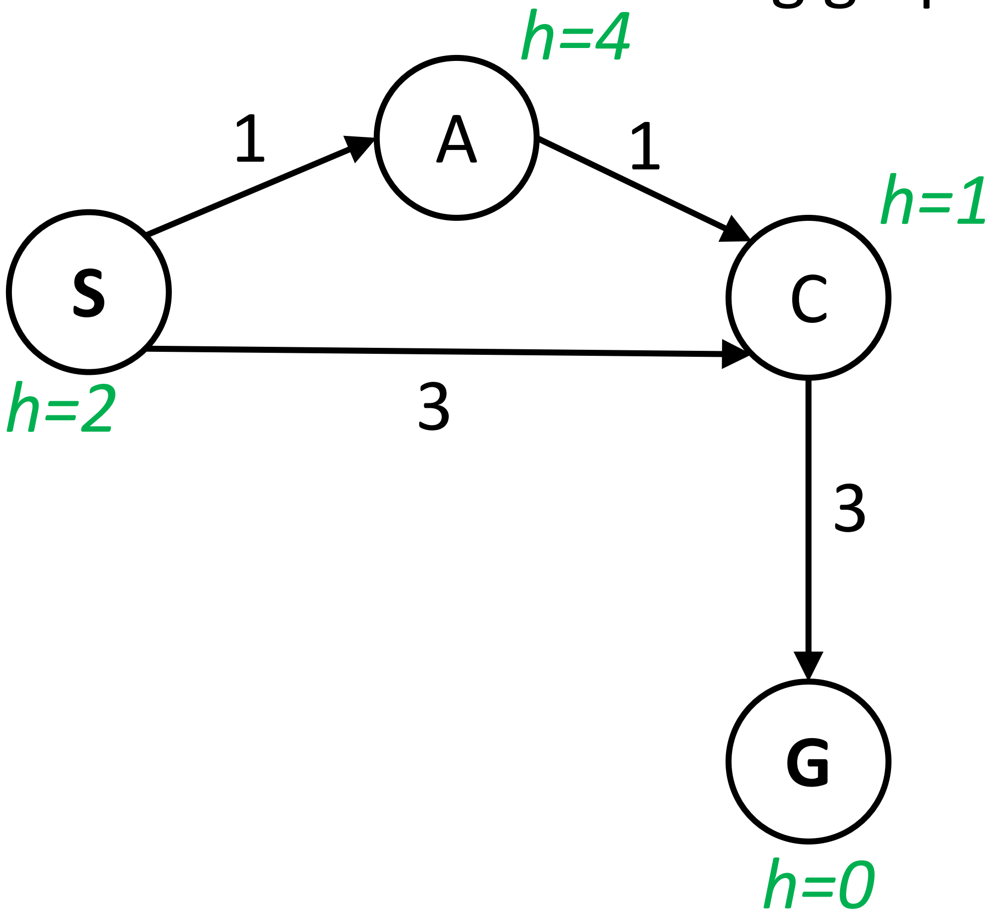
B) ~~S~~, ~~S-A~~, S-C, ~~S-A-C~~, S-C-G

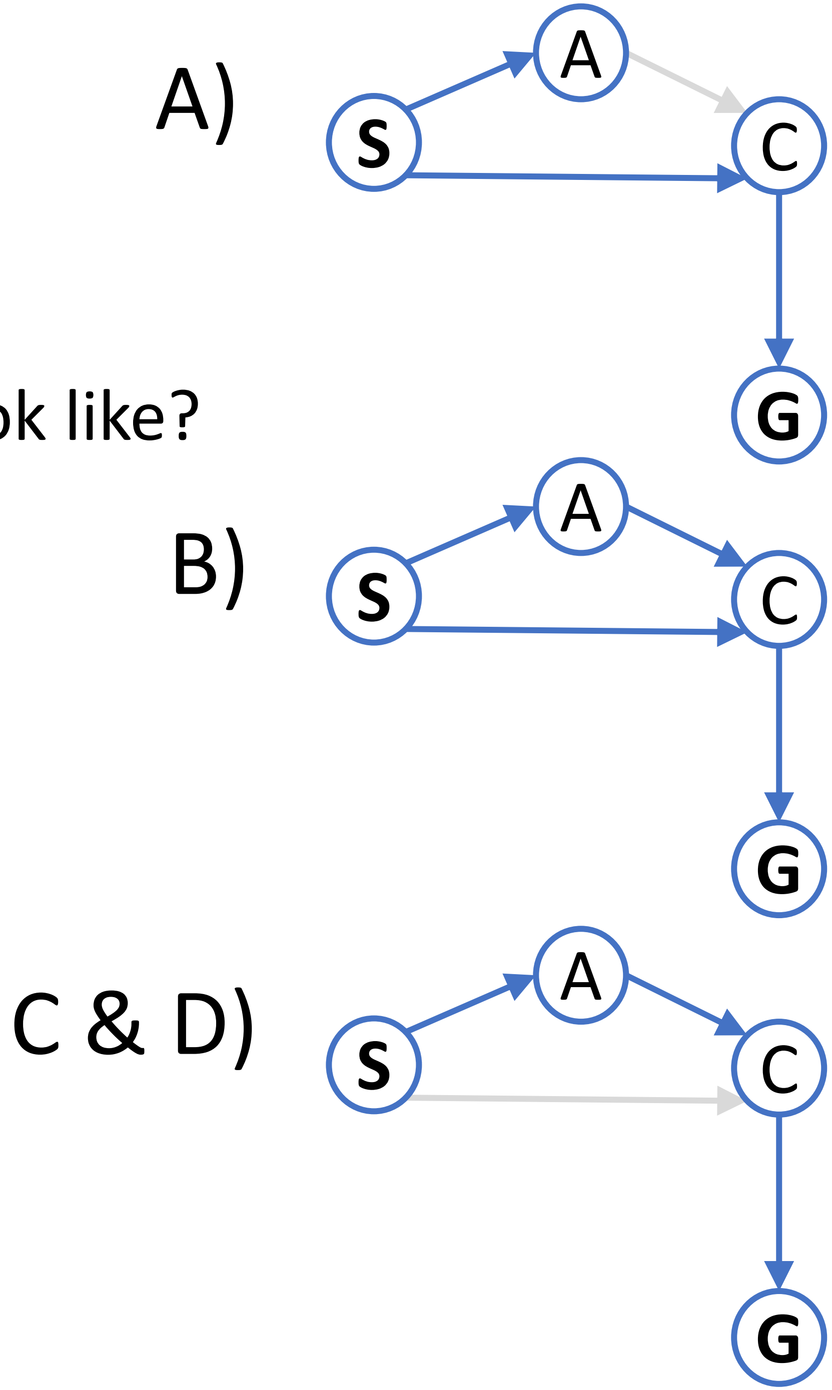
C) ~~S~~, ~~S-A~~, ~~S-A-C~~, S-A-C-G

D) ~~S~~, ~~S-A~~, ~~S-C~~, ~~S-A-C~~, S-A-C-G

Quiz: A* Graph Search 2

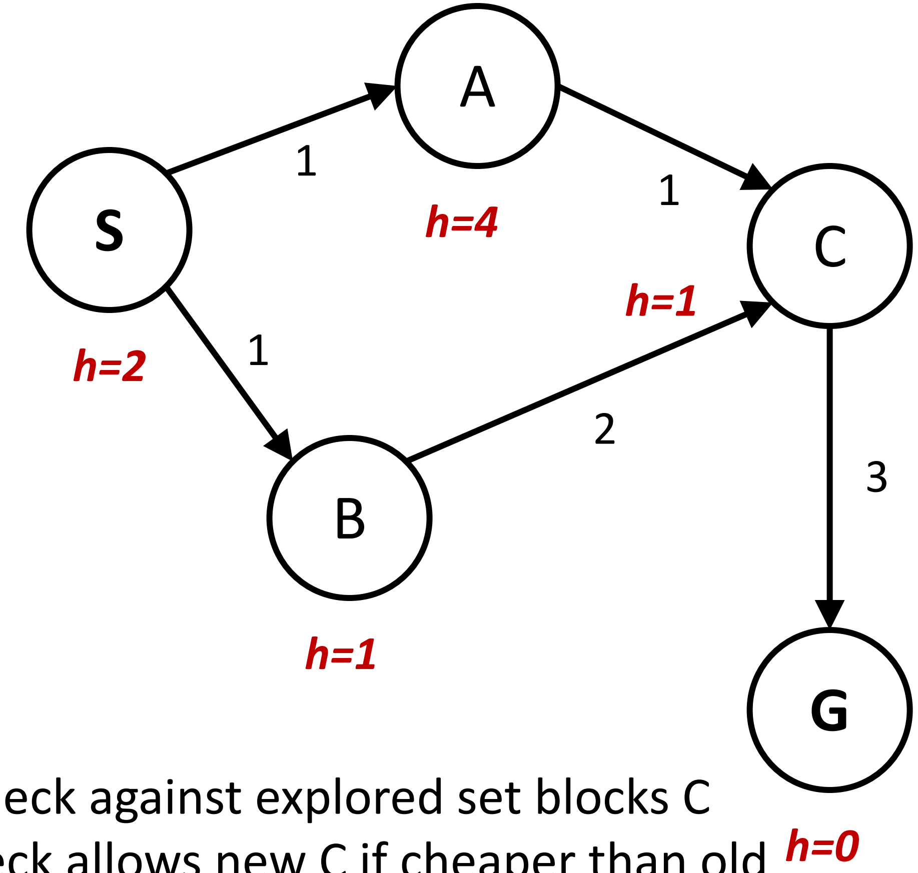
- What does the resulting graph tree look like?



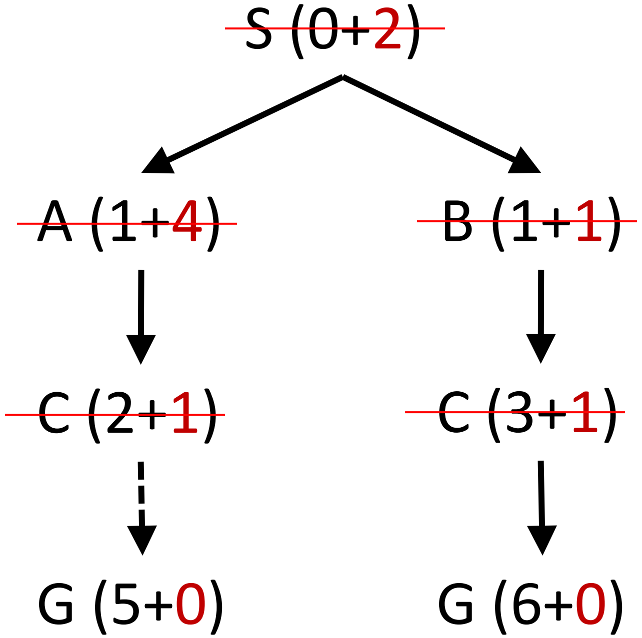


A* Graph Search Gone Wrong?

State space graph



Search tree



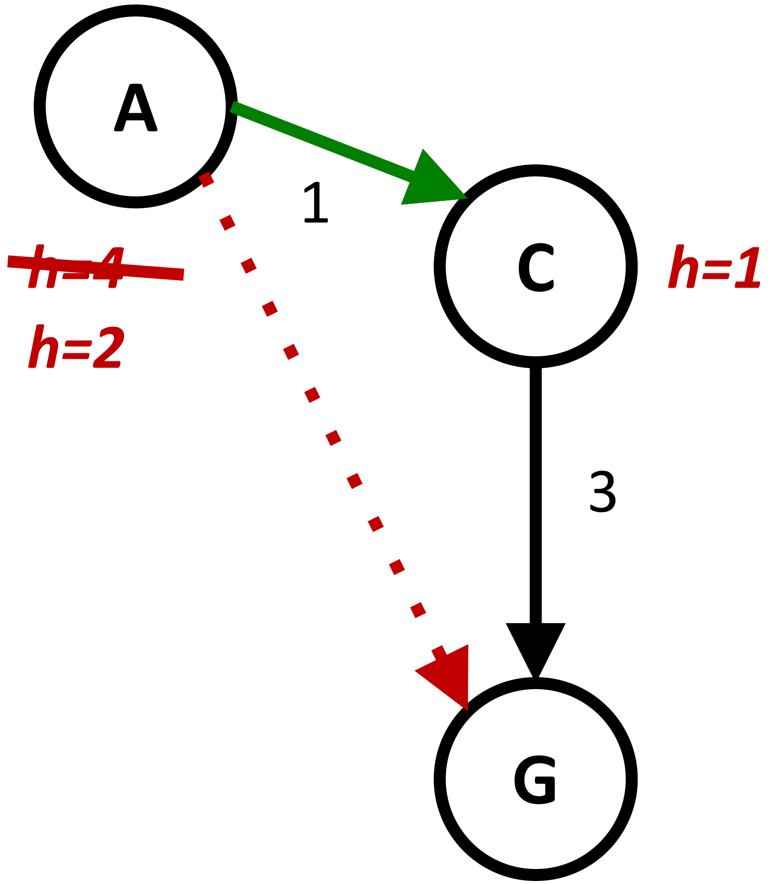
-Simple check against explored set blocks C

-Fancy check allows new C if cheaper than old but requires recalculating C's descendants

Explored Set: S B C A

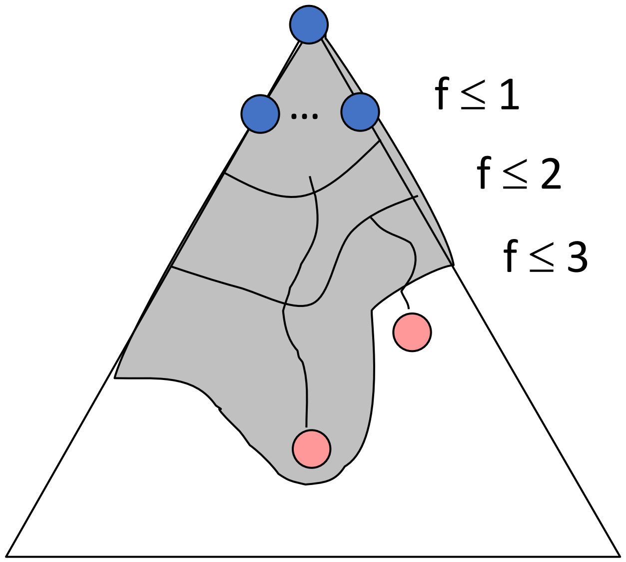
Consistency of Heuristics

- Main idea: estimated heuristic costs ≤ actual costs
  - Admissibility: heuristic cost ≤ actual cost to goal
    - h(A) ≤ actual cost from A to G
  - Consistency: heuristic "arc" cost ≤ actual cost for each arc
    - h(A) – h(C) ≤ cost(A to C)
    - triangle inequality: h(A) ≤ c(A-C) + h(C)
- Consequences of consistency:
  - The f value along a path never decreases
    - h(A) ≤ cost(A to C) + h(C)
  - A* graph search is optimal



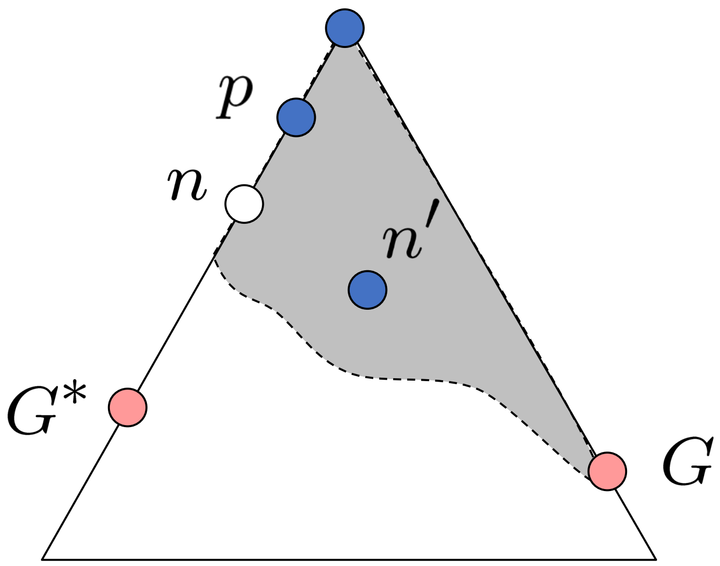
Optimality of A* Graph Search

- Sketch: consider what A* does with a consistent heuristic:
  - Fact 1: In tree search, A* expands nodes in increasing total f value (f-contours)
  - Fact 2: For every state s, nodes that reach s optimally are expanded before nodes that reach s suboptimally
  - Result: A* graph search is optimal



Optimality of A* Graph Search: Proof

- For any n on path to G^*, let n' be a worse node for the same state
- Let p be the ancestor of n that was on the queue when n' was added in the queue
- Claim: p will be expanded before n'
  - $f(p) \leq f(n)$ because of consistency
  - $f(n) < f(n')$ because n' is suboptimal
  - p would have been expanded before n'
- Thus n will be expanded before n'
- All ancestors of G^* are not blocked



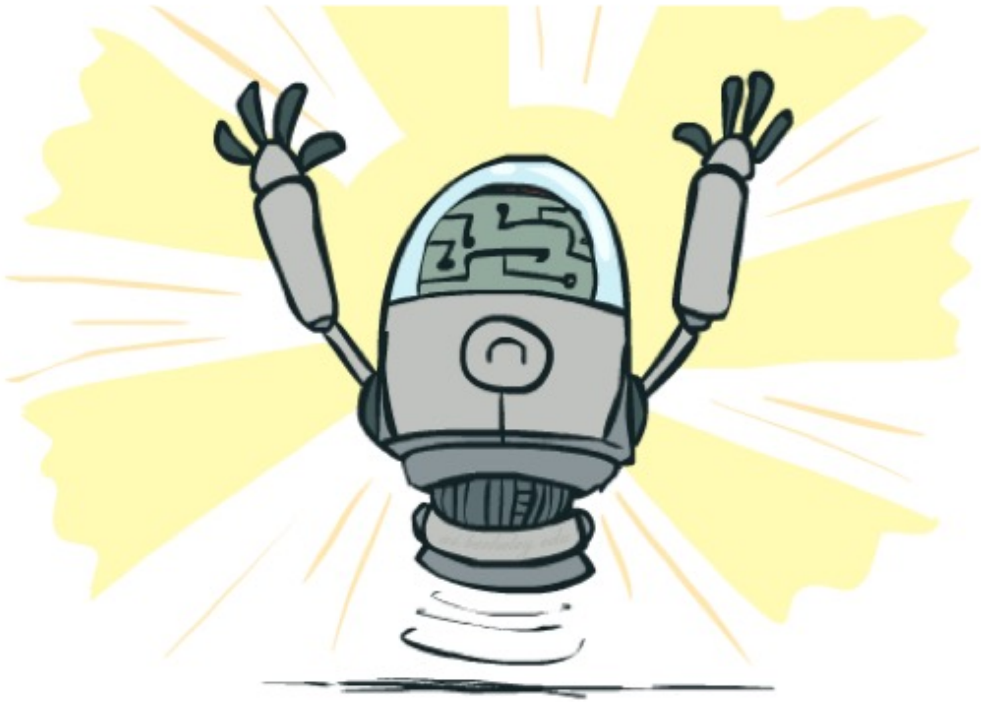
Optimality of A* Search

- Tree search:
  - A* is optimal if heuristic is admissible
  - UCS is a special case (h = 0)

- Graph search:
  - A* optimal if heuristic is consistent
  - UCS optimal (h = 0 is consistent)

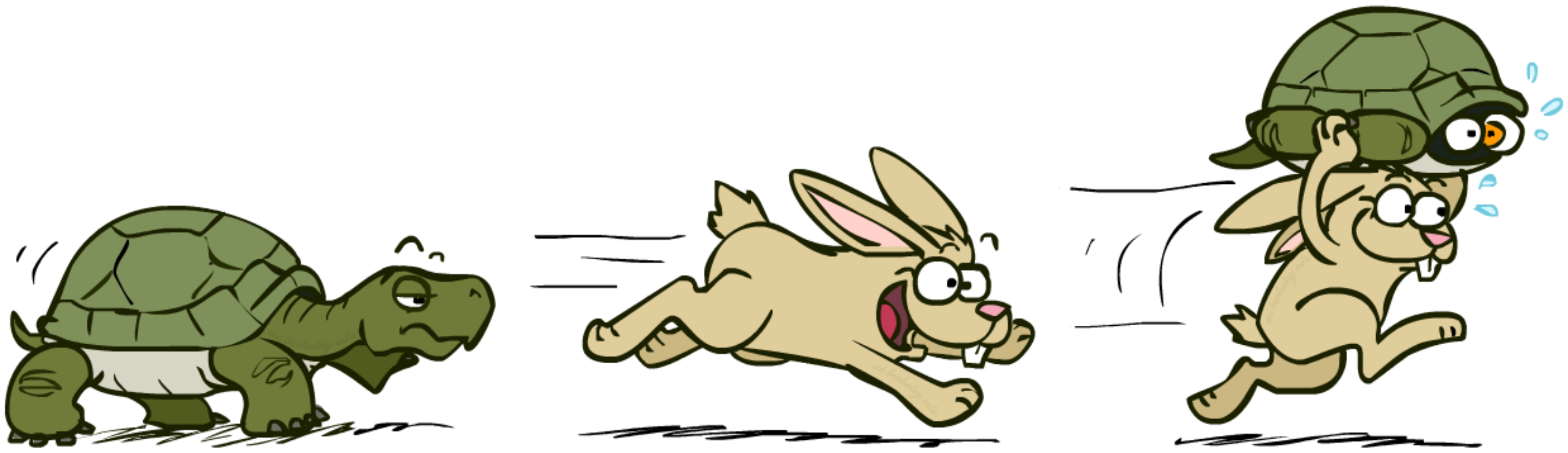
- Consistency implies admissibility

- In general, most natural admissible heuristics tend to be consistent, especially if from relaxed problems



Summary of A*

- A* uses both backward costs and (estimates of) forward costs
- A* is optimal with admissible / consistent heuristics
- Heuristic design is key: often use relaxed problems



Summary

Shuai Li
https://shuaili8.github.io

- Rational agents
- Search problems
- Uninformed Search Methods
  - Depth-First Search
  - Breadth-First Search
  - Uniform-Cost Search
- Informed Search Methods
  - Heuristics
  - Greedy Search
  - A* Search
  - Graph Search

Questions?