Lecture 6: More on Connectivity

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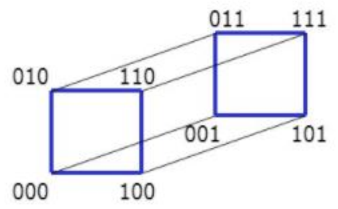
Vertex cut set and connectivity

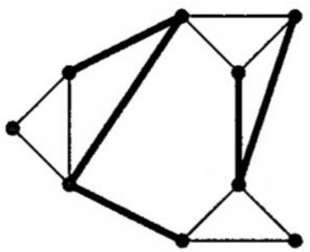
- A proper subset S of vertices is a vertex cut set if the graph $G - S$ is disconnected
- The connectivity, $\kappa(G)$, is the minimum size of a vertex set S of G such that $G - S$ is disconnected or has only one vertex
  - The graph is k-connected if $k \leq \kappa(G)$
- $\kappa(K_n) := n - 1$
- If G is disconnected, $\kappa(G) = 0$
  - $\Rightarrow$ A graph is connected $\Leftrightarrow \kappa(G) \geq 1$
- If G is connected, non-complete graph of order n, then
$$1 \leq \kappa(G) \leq n - 2$$



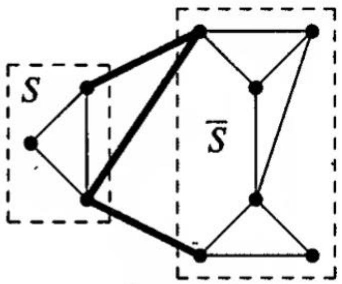
Q_3

- For convention, $\kappa(K_1) = 0$
- Example (4.1.3, W) For k-dimensional cube $Q_k = \{0,1\}^k$, $\kappa(Q_k) = k$

Edge-connectivity



disconnecting set



edge cut

- A disconnecting set of edges is a set $F \subseteq E(G)$ such that $G - F$ has more than one component
  - A graph is k-edge-connected if every disconnecting set has at least k edges
  - The edge-connectivity of G, written $\lambda(G)$, is the minimum size of a disconnecting set
- Given $S, T \subseteq V(G)$, we write $[S, T]$ for the set of edges having one endpoint in S and the other in T
  - An edge cut is an edge set of the form $[S, S^c]$ where S is a nonempty proper subset of $V(G)$
- Every edge cut is a disconnecting set, but not vice versa
- Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Connectivity and edge-connectivity

- Proposition (1.4.2, D) If G is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$
- If $\delta(G) \geq n-2$, then $\kappa(G) = \delta(G)$

  that is $\kappa(G) = \lambda(G) = \delta(G)$

- Theorem (4.1.11, W) If G is a 3-regular graph, then $\kappa(G) = \lambda(G)$

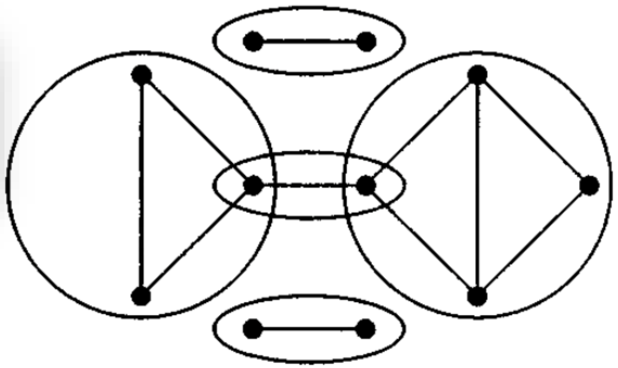
Properties of edge cut

- When $\lambda(G) < \delta(G)$, a minimum edge cut cannot isolate a vertex
- Similarly for (any) edge cut
- Proposition (4.1.12, W) If S is a set of vertices in a graph G, then
$$|[S, S^c]| = \sum_{v \in S} d(v) - 2e(G[S])$$
- Corollary (4.1.13, W) If G is a simple graph and $|[S, S^c]| < \delta(G)$, then $|S| > \delta(G)$
    - $|S|$ must be much larger than a single vertex

Blocks

- A block of a graph G is a maximal connected subgraph of G that has no cut-vertex. If G itself is connected and has no cut-vertex, then G is a block
- Example

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G
- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G



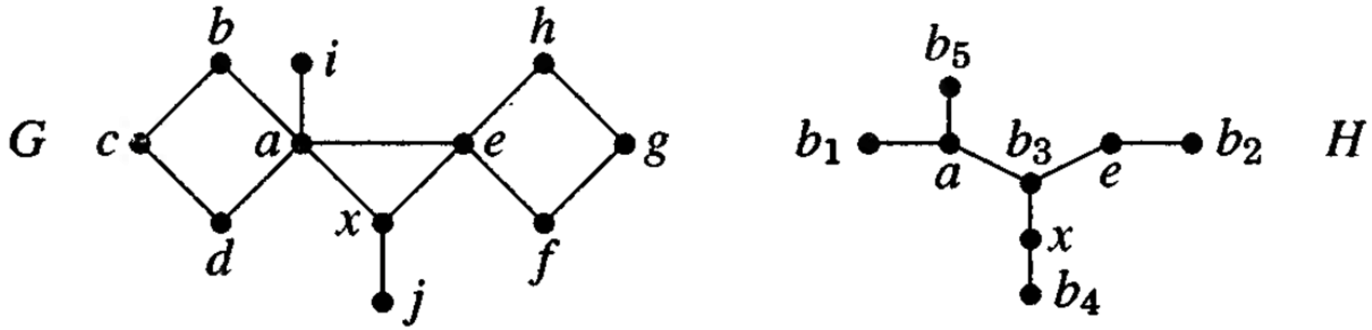
- An edge of a cycle cannot itself be a block
  - An edge is block $\Leftrightarrow$ it is a bridge
  - The blocks of a tree are its edges
- If a block has more than two vertices, then it is 2-connected
  - The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

Intersection of two blocks

- Proposition (4.1.19, W) Two blocks in a graph share at most one vertex
  - When two blocks share a vertex, it must be a cut-vertex
- Every edge is a subgraph with no cut-vertex and hence is in a block. Thus blocks in a graph decompose the edge set

Block-cutpoint graph

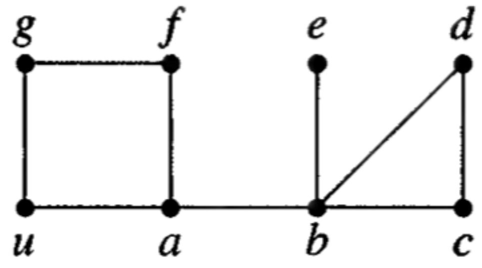
- The block-cutpoint graph of a graph G is a bipartite graph H in which one partite set consists of the cut-vertices of G, and the other has a vertex b_i for each block B_i of G. We include vb_i as an edge of H $\Leftrightarrow$ $v \in B_i$



- (Ex34, S4.1, W) When G is connected, its block-cutpoint graph is a tree

Depth-first search (DFS)

- Depth-first search

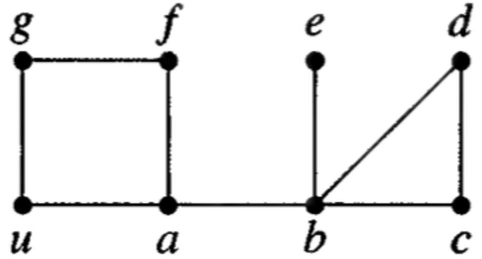


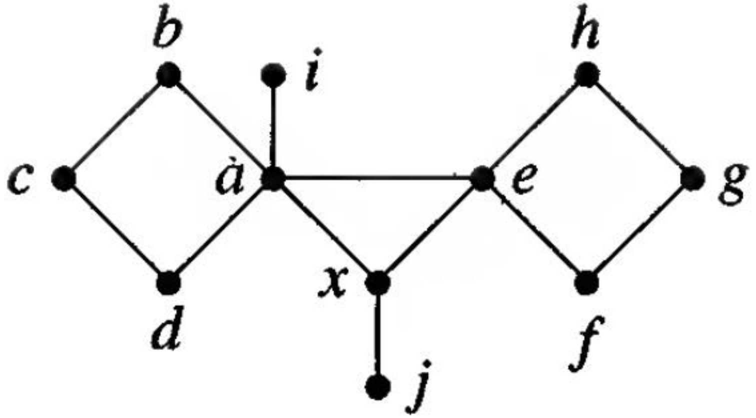
- Lemma (4.1.22, W) If T is a spanning tree of a connected graph grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T

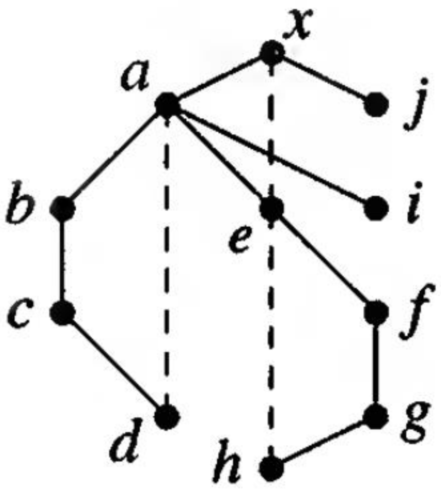
Finding blocks by DFS

- **Input**: A connected graph G
- **Idea**: Build a DFS tree T of G, discarding portions of T as blocks are identified. Maintain one vertex called ACTIVE
- **Initialization**: Pick a root $x \in V(H)$; make x ACTIVE; set $T = \{x\}$
- **Iteration**: Let v denote the current active vertex
  - If v has an unexplored incident edge vw, then
    - If $w \notin V(T)$, then add vw to T, mark vw explored, make w ACTIVE
    - If $w \in V(T)$, then w is an ancestor of v; mark vw explored
  - If v has no more unexplored incident edges, then
    - If $v \neq x$ and w is a parent of v, make w ACTIVE. If no vertex in the current subtree T' rooted at v has an explored edge to an ancestor above w, then $V(T') \cup \{w\}$ is the vertex set of a block; record this information and delete $V(T')$
    - if $v = x$, terminate

Example







Strong orientation

- Theorem (2.5, L; 4.2.14, W; Robbins 1939) A graph has a strong orientation, i.e. an orientation that is a strongly connected digraph $\Leftrightarrow$ it is 2-edge-connected
  - A directed graph is strongly connected if for every pair of vertices (v, w), there is a directed path from v to w
  - Proposition (2.4, L) Let $xy \in T$ which is not a bridge in G and x is a parent of y. Then there exists an edge in G but not in T joining some descendant a of y and some ancestor b of x

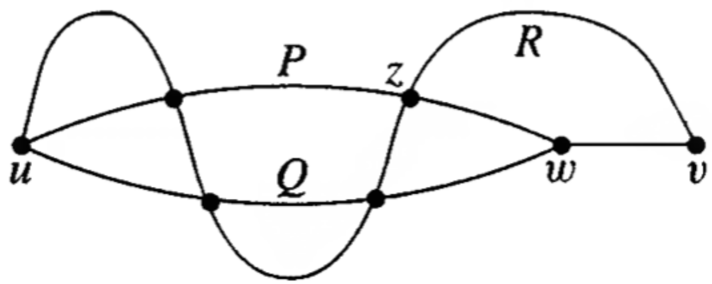
- The blocks of a loopless graph are its isolated vertices, bridges, and its maximal 2-connected subgraphs

Lemma (4.1.22, W) If T is a spanning tree of a connected graph grown by DFS from u, then every edge of G not in T consists of two vertices v, w such that v lies on the u, w-path in T

2-Connected Graphs

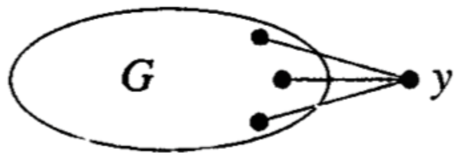
2-connected graphs

- Two paths from u to v are internally disjoint if they have no common internal vertex
- Theorem (4.2.2, W; Whitney 1932)
  A graph G having at least three vertices is 2-connected $\Longleftrightarrow$ for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G

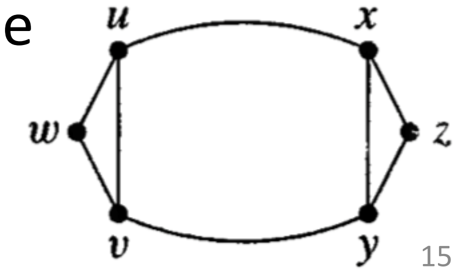


Equivalent definitions for 2-connected graphs

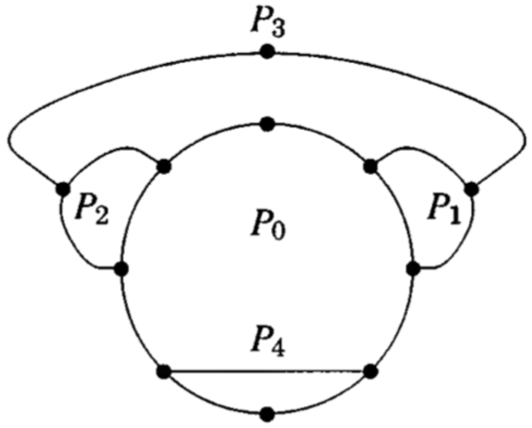
- Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



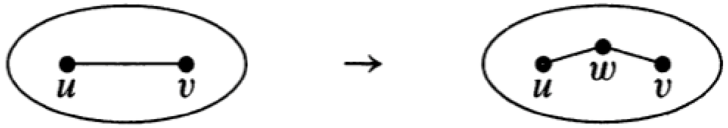
- Theorem (4.2.4, W) For a graph G with at least three vertices, TFAE
  - G is connected and has no cut-vertex
  - For all $x, y \in V(G)$, there are internally disjoint x, y-paths
  - For all $x, y \in V(G)$, there is a cycle through x and y
  - $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle



Ear decomposition

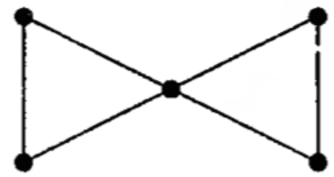


- An ear of a graph G is a maximal path whose internal vertices have degree 2 in G
- An ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$
- Theorem (4.2.8, W)
  A graph is 2-connected $\Leftrightarrow$ it has an ear decomposition.
  Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition
  - Corollary (4.2.6, W) If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies G has at least one cycle

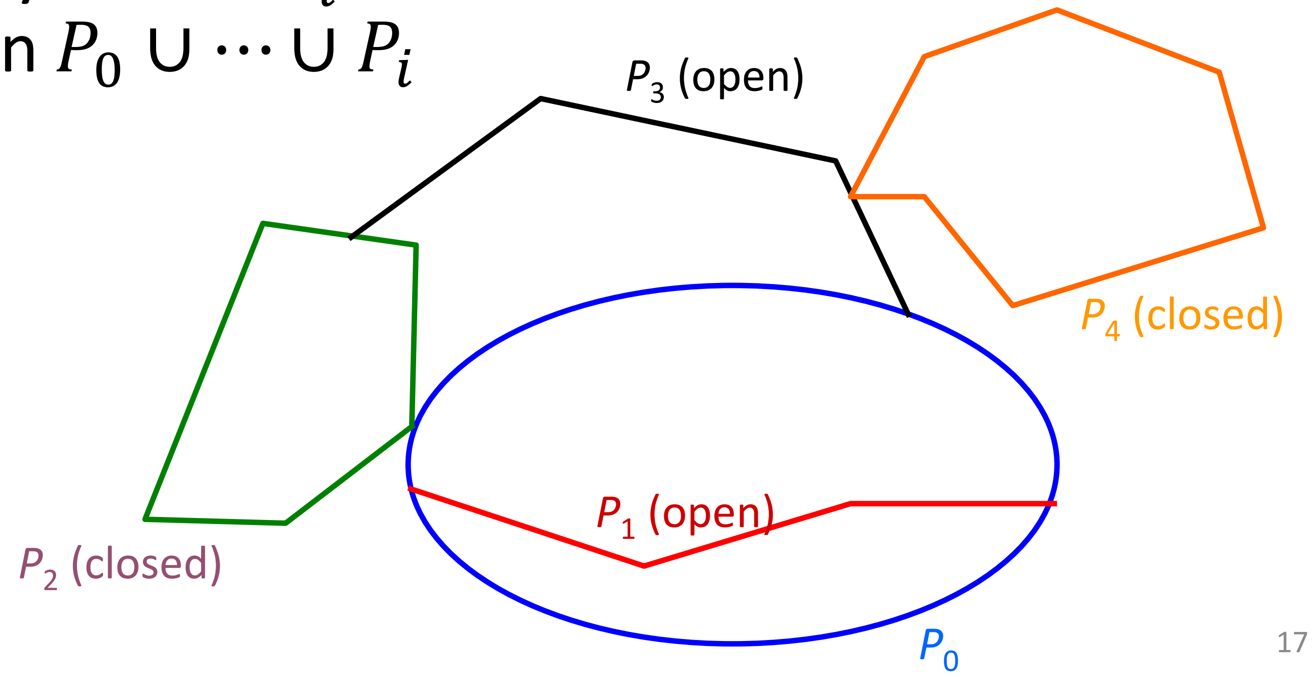


Closed-ear

- A closed ear of a graph G is a cycle C such that all vertices of C except one have degree 2 in G



- A closed-ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an (open) ear or a closed ear in $P_0 \cup \cdots \cup P_i$



Closed-ear decomposition

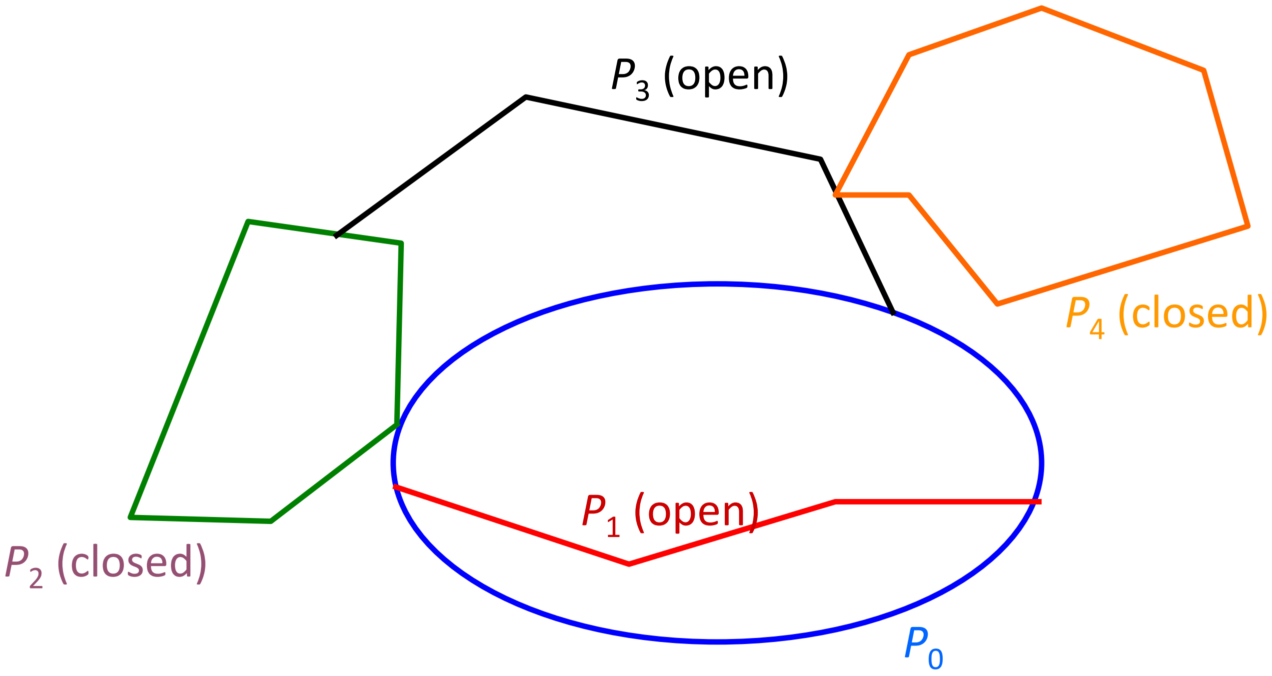
- Theorem (4.2.10, W)
  A graph is 2-edge-connected $\Leftrightarrow$ it has a closed-ear decomposition.
  Every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition

Proposition (1.2.14, W)
An edge e is a bridge $\Leftrightarrow$ e lies on no cycle of G

- Or equivalently, an edge e is not a bridge $\Leftrightarrow$ e lies on a cycle of G

Strong orientation (Revisited)

- Theorem (2.5, L; 4.2.14, W; Robbins 1939) A graph has a strong orientation, i.e. an orientation that is a strongly connected digraph ⟺ it is 2-edge-connected



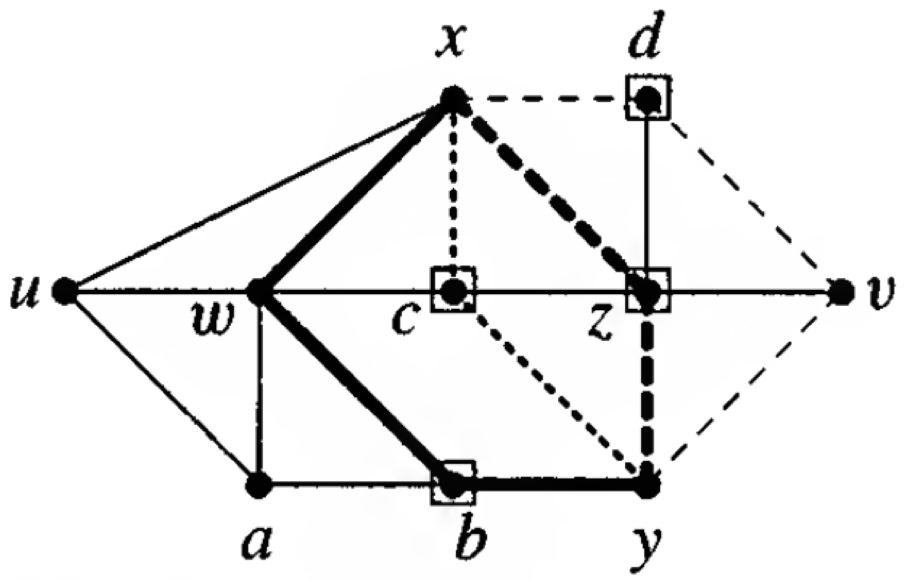
k-Connected and k-Edge-Connected graphs

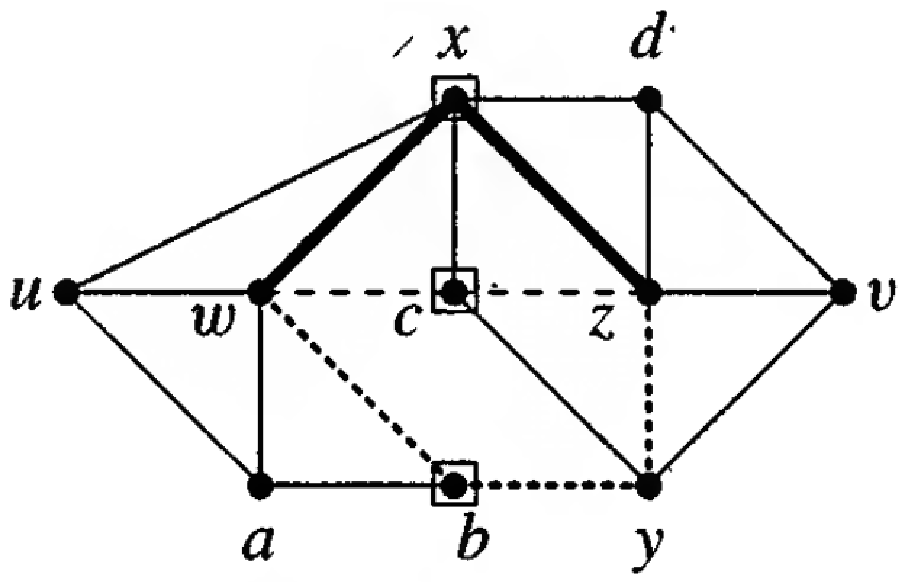
x, y-cut

- Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path
  - Let $\kappa(x, y)$ be the minimum size of an x, y-cut
  - Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths
  - $\kappa(x, y) \geq \lambda(x, y)$
- For $X, Y \subseteq V(G)$, an X, Y-path is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$

Example (4.2.16, W)

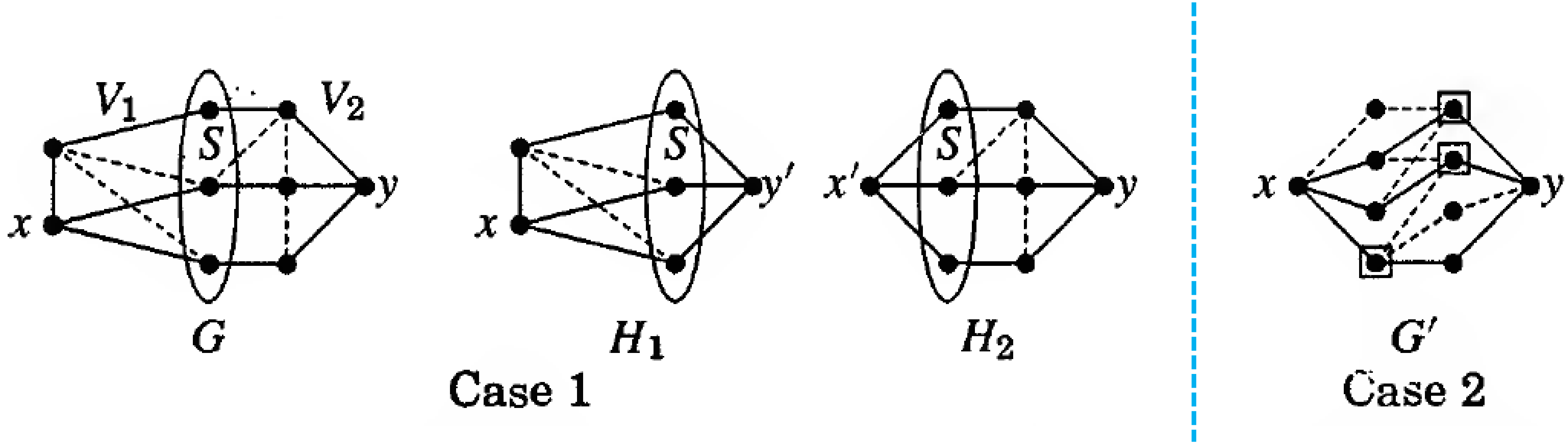
- $S = \{b, c, z, d\}$
- $\kappa(x, y) = \lambda(x, y) = 4$
- $\kappa(w, z) = \lambda(w, z) = 3$





Menger's Theorem

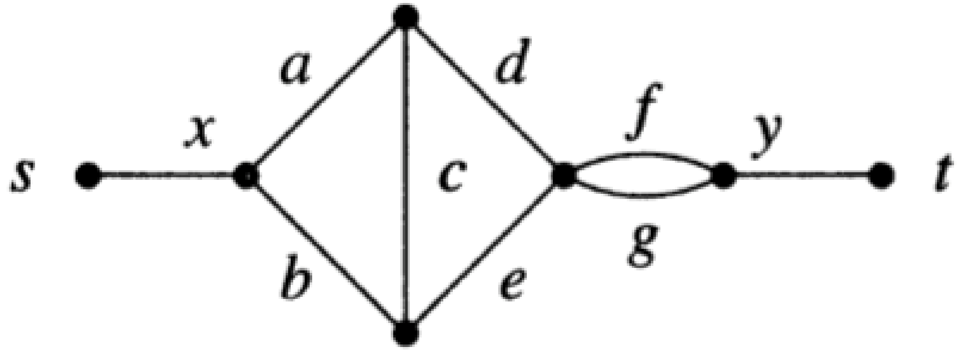
- Theorem (4.2.17, W; 3.3.1, D; Menger, 1927) If x, y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$

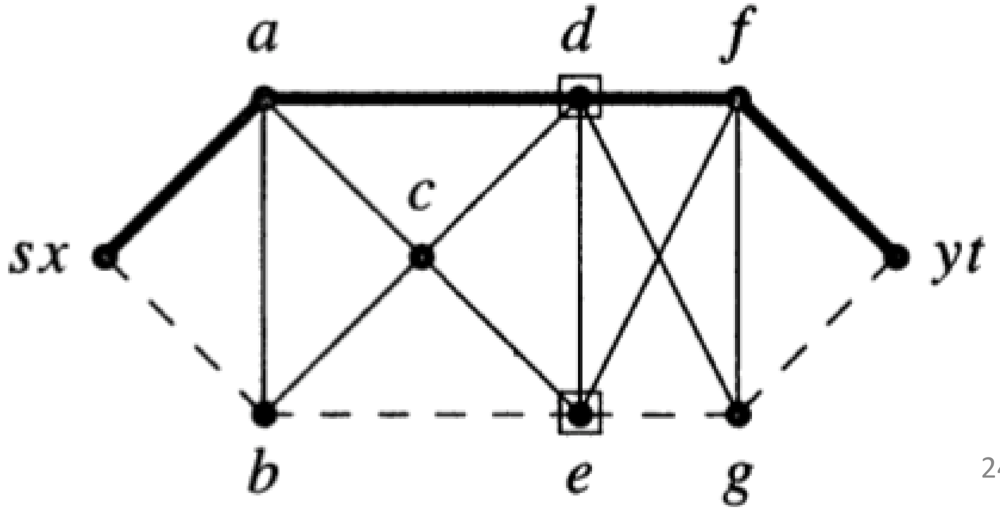


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Edge version

- Theorem (4.2.19, W) If x and y are distinct vertices of a graph G, then the minimum size $\kappa'(x, y)$ of an x, y-disconnecting set of edges equals the maximum number $\lambda'(x, y)$ of pairwise edge-disjoint x, y-paths
  - The line graph $L(G)$ of a graph G is the graph whose vertices are the edges of G with $ef \in E(L(G))$ when $e = uv$ and $f = vw$ in G



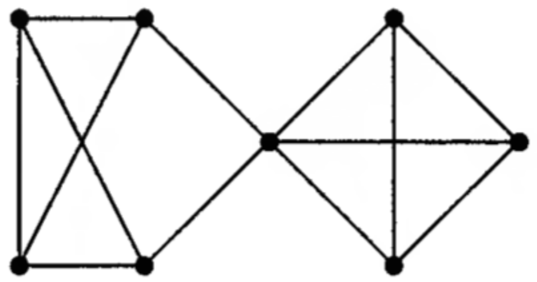


Back to connectivity

- Theorem (4.2.21, W)

$$\kappa(G) = \min_{x \neq y \in V(G)} \lambda(x, y), \qquad \lambda(G) = \min_{x \neq y \in V(G)} \lambda'(x, y)$$

  - Lemma (4.2.20, W) Deletion of an edge reduces connectivity by at most 1



Application of Menger’s Theorem

CSDR

- Let $\boldsymbol{A} = A_1, \dots, A_m$ and $\boldsymbol{B} = B_1, \dots, B_m$ be two family of sets. A common system of distinct representatives (CSDR) is a set of m elements that is both an system of distinct representatives (SDR) for $\boldsymbol{A}$ and an SDR for $\boldsymbol{B}$

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$S_1 = \{2, 8\},$$
$$S_2 = \{8\},$$
$$S_3 = \{5, 7\},$$
$$S_4 = \{2, 4, 8\},$$
$$S_5 = \{2, 4\}.$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

Equivalent condition for CSDR

- Theorem (4.2.25, W; Ford-Fulkerson 1958) Families $\boldsymbol{A} = \{A_1, \dots, A_m\}$ and $\boldsymbol{B} = \{B_1, \dots, B_m\}$ have a common system of distinct representatives (CSDR) $\Leftrightarrow$
$$\left|\left(\bigcup_{i \in I} A_i\right) \cap \left(\bigcup_{j \in J} B_j\right)\right| \geq |I| + |J| - m$$
for every pair $I, J \subseteq [m]$

Summary

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- Connectivity, edge-connectivity
- Blocks, block-cutpoint graph, DFS
- 2-connectivity
  - Equivalent definitions for 2-connected graphs
  - (Closed) Ear decomposition
- k-connectivity
  - Menger's Theorem, for $xy \notin E(G)$, $\kappa(x,y) = \lambda(x,y)$
  - $\kappa'(x,y) = \lambda'(x,y)$
  - $\kappa(G) = \min_{x \neq y \in V(G)} \lambda(x,y)$, $\lambda(G) = \min_{x \neq y \in V(G)} \lambda'(x,y)$
  - Application: CSDR

Questions?