# Lecture 7: Coloring

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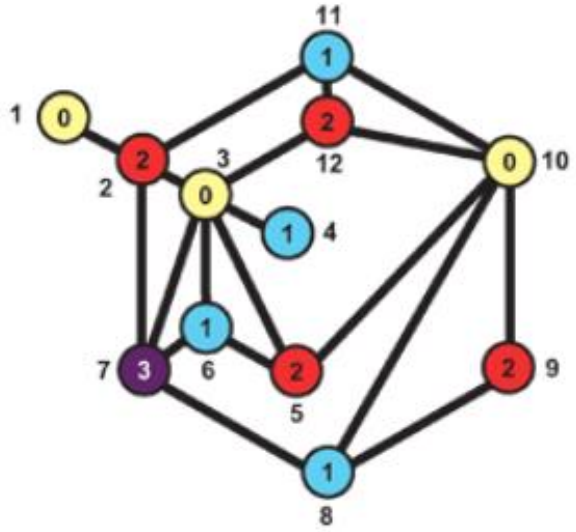
https://shuaili8.github.io/Teaching/CS3330/index.html

# Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



# Definitions

- Given a graph $G$ and a positive integer $k$, a $k$-coloring is a function $K: V(G) \longrightarrow \{1, \dots, k\}$ from the vertex set into the set of positive integers less than or equal to $k$. If we think of the latter set as a set of $k$ "colors," then $K$ is an assignment of one color to each vertex.
- We say that $K$ is a proper $k$-coloring of $G$ if for every pair $u, v$ of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph $G$, we say that $G$ is $k$-colorable
- In a proper coloring, each color class is an independent set. Then $G$ is $k$-colorable $\Leftrightarrow V(G)$ is the union of $k$ independent sets

# Chromatic number

- Given a graph $G$, the chromatic number of $G$, denoted by $\chi(G)$, is the smallest integer $k$ such that $G$ is $k$-colorable. $G$ is said to be $k$-chromatic
- Examples

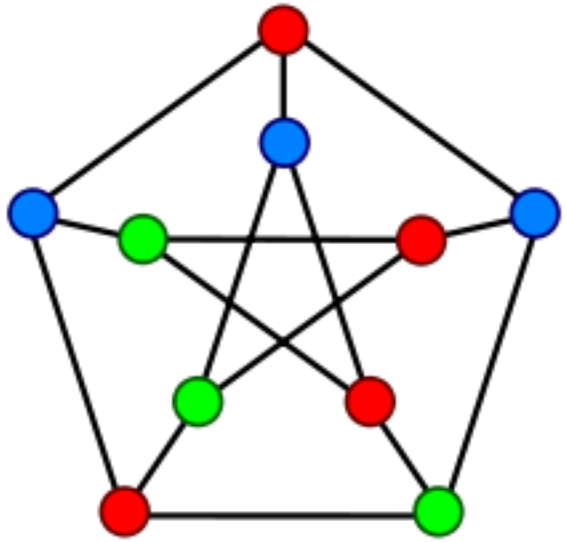
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$$\chi(E_n) = 1, \quad \leftarrow \text{Empty graph}$$

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph $G$ of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

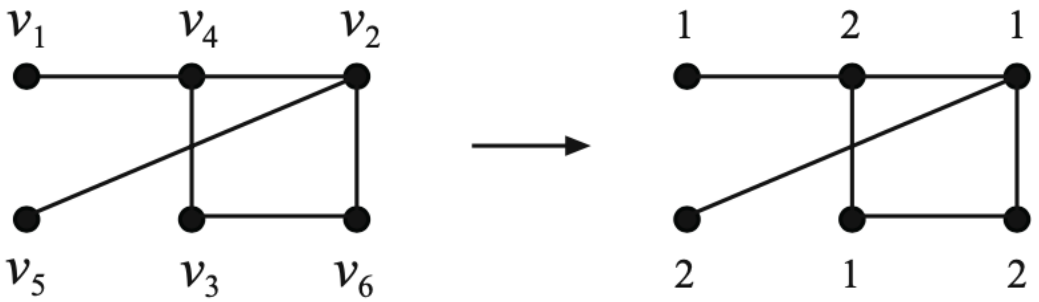
# Bounds on Chromatic number

- Theorem (1.41, H) For any graph $G$ of order $n$, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

# Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \ldots, v_n$
- Next, order the available colors $(1,2, \ldots, n)$ in some way
  - Start coloring by assigning color $1$ to vertex $v_1$
  - If $v_1$ and $v_2$ are adjacent, assign color $2$ to vertex $v_2$; otherwise, use color $1$
  - To color vertex $v_i$, use the first available color that has not been used for any of $v_i$'s previously colored neighbors

# Examples: Different orders result in different number of colors

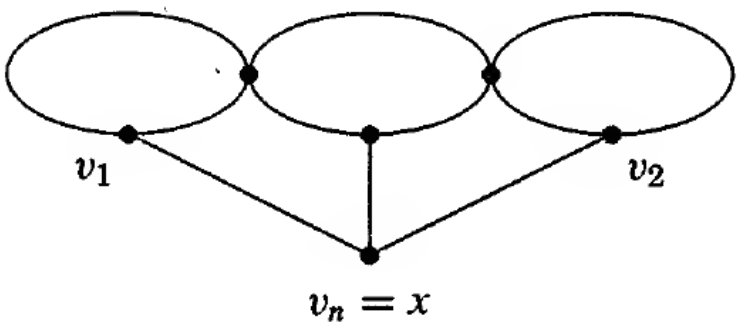



# Bound using the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
  The equality is obtained for complete graphs and odd cycles

# Brooks's theorem

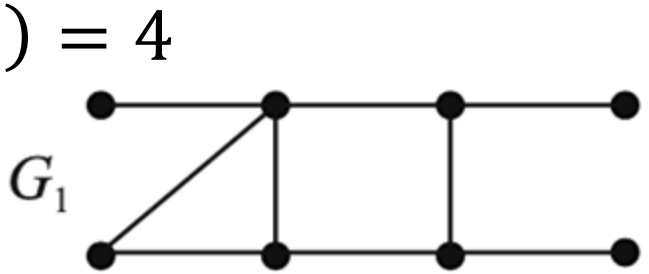
- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$

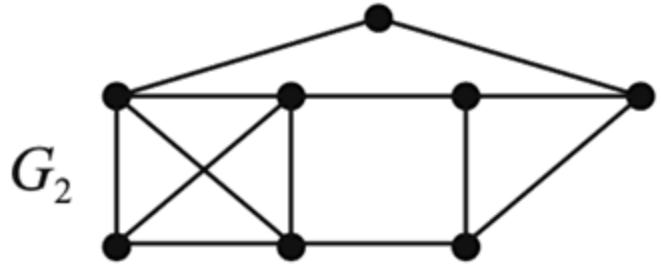



- ⇒The Petersen graph is 3-colorable

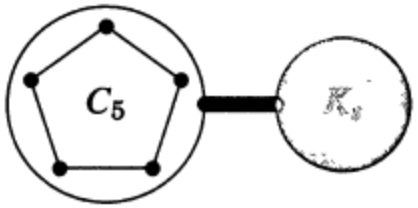
# Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of $G$
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$




- Theorem (1.44, H; 5.1.7, W) For any graph $G$, $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



# Chromatic number and independence number

- Theorem (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph $G$ of order $n$,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The independence number of a graph $G$, denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then $G$ is $k$-colorable $\Leftrightarrow V(G)$ is the union of $k$ independent sets

# Extremal properties for $k$-chromatic graphs

- Proposition (5.2.5, W) Every $k$-chromatic graph with $n$ vertices has at least $\binom{k}{2}$ edges
  - Equality holds for a complete graph plus isolated vertices.

> In a proper coloring, each color class is an independent set. Then $G$ is $k$-colorable $\Leftrightarrow$ $V(G)$ is the union of $k$ independent sets

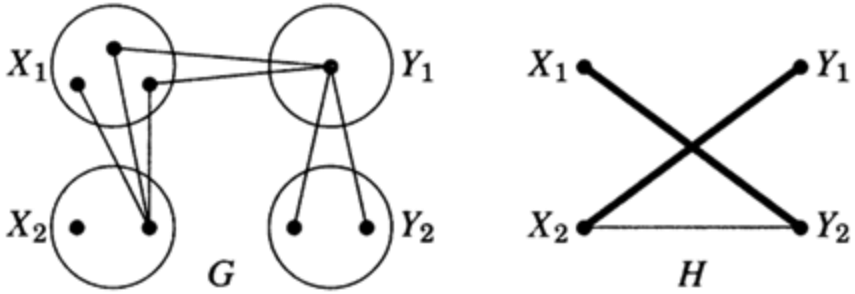
- The Turán graph $T_{n,r}$ is the complete $r$-partite graph with $n$ vertices whose partite sets differ by at most 1 vertex
  - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- Lemma (5.2.8, W) Among simple $r$-partite (that is, $r$-colorable) graphs with $n$ vertices, the Turán graph is the unique graph with the most edges
- Turán's Theorem (5.2.9, W; Turán 1941) Among the $n$-vertex simple $K_{r+1}$-free graphs, $T_{n,r}$ has the maximum number of edges

# Color-critical

- If $\chi(H) < \chi(G) = k$ for every proper subgraph $H$, then $G$ is color-critical or $k$-critical
- $K_2$ is the only 2-critical graph
  $K_1$ is the only 1-critical graph
- (5.2.12, W) A graph with no isolated vertices is color-critical $\Leftrightarrow$ $\chi(G - e) < \chi(G)$ for every edge $e \in E(G)$
- Proposition (5.2.13, W) Let $G$ be a $k$-critical graph
  (a) For every $v \in V(G)$, there is a proper coloring such that $v$ has a unique color and other $k - 1$ colors all appear on $N(v)$
  $\Rightarrow \delta(G) \geq k - 1$
  (b) For every $e \in E(G)$, every proper $(k - 1)$-coloring of $G - e$ gives the same color to the two endpoints of $e$

# Color-critical has edge-connectivity

- Theorem (5.2.16, W; Dirac 1953) Every $k$-critical graph is $(k-1)$-edge-connected
- Lemma (5.2.15, W; Kainen) Let $G$ be a graph with $\chi(G) > k$ and let $X, Y$ be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are $k$-colorable, then the edge cut $[X, Y]$ has at least $k$ edges

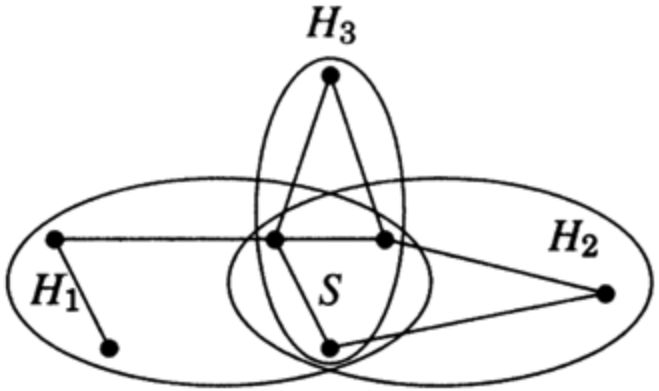



Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let $G$ be a bipartite graph. The maximum size of a matching in $G$ is equal to the minimum size of a vertex cover of its edges

Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

# Color-critical and vertex cut set

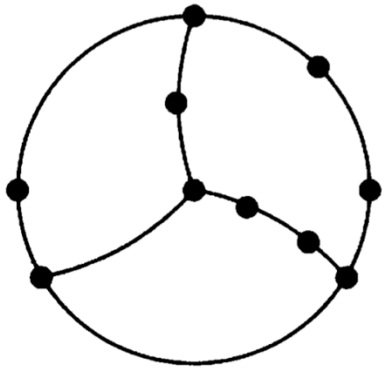
- Let $S$ be a set of vertices in a graph $G$. An $S$-lobe of $G$ is an induced subgraph of $G$ whose vertex set consists of $S$ and the vertices of a component in $G - S$



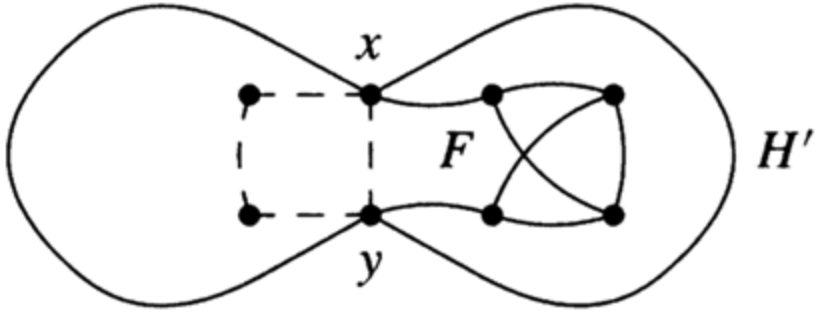

- Proposition (5.2.18, W) If $G$ is $k$-critical, then $G$ has no clique cutset. In particular, if $G$ has a cutset $S = \{x, y\}$, then $x, y$ are non-adjacent and $G$ has an $S$-lobe $H$ such that $\chi(H + xy) = k$

# Chromatic number $4$ has a $K_4$-subdivision

- Theorem (5.2.20, W; Dirac 1952) Every graph with chromatic number at least $4$ contains a $K_4$-subdivision



a subdivision of $K_4$

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Lemma (4.2.3, W; Expansion Lemma) If $G$ is a $k$-connected graph, and $G'$ is obtained from $G$ by adding a new vertex $y$ with at least $k$ neighbors in $G$, then $G'$ is $k$-connected

# Hajós' conjecture

- Hajós' conjecture [1961]: Every $k$-chromatic graph contains a subdivision of $K_k$
- $k = 2$: Every $2$-chromatic graph has a nontrivial path
- $k = 3$: Every $3$-chromatic graph has a cycle
- It is open for $k = 5,6$
- Exercise (Ex5.2.40, W) It is false for $k = 7$ or $8$

# Chromatic Polynomials

# Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $\chi(G;k)$ to be the number of different colorings of a graph $G$ using at most $k$ colors
- Examples:
  - How many different colorings of $K_4$ using $4$ colors?
    - $4 \times 3 \times 2 \times 1$
    - $\chi(K_4;4) = 24$
  - How many different colorings of $K_4$ using $6$ colors?
    - $6 \times 5 \times 4 \times 3$
    - $\chi(K_4;6) = 360$
  - How many different colorings of $K_4$ using $2$ colors?
    - $0$
    - $\chi(K_4;2) = 0$

# Examples

- If $k \geq n$

$$\chi(K_n; k) = k(k-1)\cdots(k-n+1)$$

- If $k < n$

$$\chi(K_n; k) = 0$$

- $G$ is $k$-colorable $\Leftrightarrow \ \chi(G) \leq k \ \Leftrightarrow \chi(G; k) > 0$
- $\chi(G) = \min\{k \geq 1 : \chi(G; k) > 0\}$

# Chromatic recurrence

- $G - e$ and $G/e$

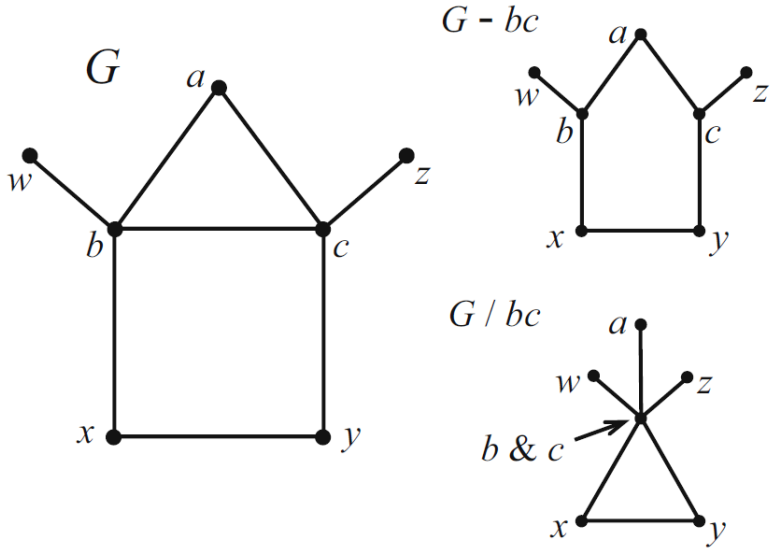



FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let $G$ be a graph and $e$ be any edge of $G$. Then

$$\chi(G;k) = \chi(G - e;k) - \chi(G/e;k)$$

# Use chromatic recurrence to compute $\chi(G;k)$

- Example: Compute $\chi(P_3;k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $\chi(P_3;1) = 0, \chi(P_3;2) = 2$


FIGURE 1.102. Two 2-colorings of $P_3$

- Example: What is $\chi(K_n - e;k)$?

# More examples

- Path $P_{n-1}$ has $n-1$ edges ($n$ vertices)

$$\chi(P_{n-1}; k) = k(k-1)^{n-1}$$

- Any tree $T$ on $n$ vertices

$$\chi(T; k) = k(k-1)^{n-1}$$

- Cycle $C_n$

$$\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$$

  - When $n$ is odd, $\chi(C_n; 2) = 0, \chi(C_n; 3) > 0$
  - When $n$ is even, $\chi(C_n; 2) > 0$

# Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let $G$ be a graph of order $n$
  - $\chi(G;k)$ is a polynomial in $k$ of degree $n$
  - The leading coefficient of $\chi(G;k)$ is $1$
  - The constant term of $\chi(G;k)$ is $0$
    - If $G$ has $i$ components, then the coefficients of $k^0, \dots, k^{i-1}$ are $0$
    - $G$ is connected $\Leftrightarrow$ the coefficient of $k$ is nonzero
  - The coefficients of $\chi(G;k)$ alternate in sign
  - The coefficient of the $k^{n-1}$ term is $-|E(G)|$
    - A graph $G$ is a tree $\Leftrightarrow \chi(G;k) = k(k-1)^{n-1}$

      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) $T$ is connected with $n-1$ edges

    - A graph $G$ is complete $\Leftrightarrow \chi(G;k) = k(k-1)\cdots(k-n+1)$

# Simplicial elimination ordering

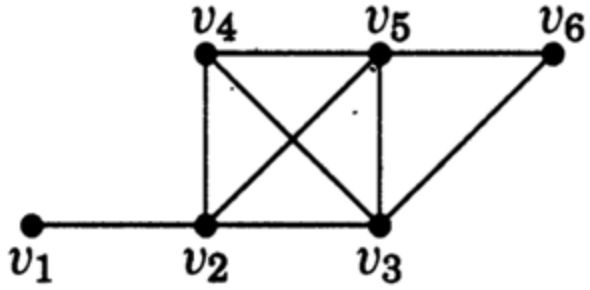
- Roots for the chromatic polynomials?
  Fundamental theorem of algebra
- A vertex of $G$ is simplicial if its neighborhood in $G$ induces a clique
- A simplicial elimination ordering is an ordering $v_n, \dots, v_1$ for deletion of vertices s.t. each vertex $v_i$ is a simplicial vertex of the graph reduced by $\{v_1, \dots, v_i\}$
- Chromatic polynomials
  If we have colored $v_1, \dots, v_{i-1}$, then there are $k - d(i)$ ways to color $v_i$ where $d(i) = |N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$. Thus

$$\chi(G; k) = \prod_{i=1}^{n} (k - d(i))$$

Nice factorization property!

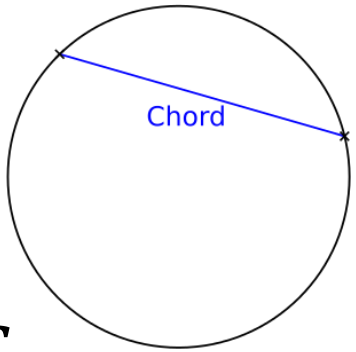
# Examples

- In a tree, a simplicial elimination ordering is a successive deletion of leaves
  - Another proof for $\chi(T;k) = k(k-1)^{n-1}$
- Example (5.3.13, W) $v_6, \ldots, v_1$ is a simplicial elimination ordering. The values $d(i)$ are 0,1,1,2,3,2. Thus the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$

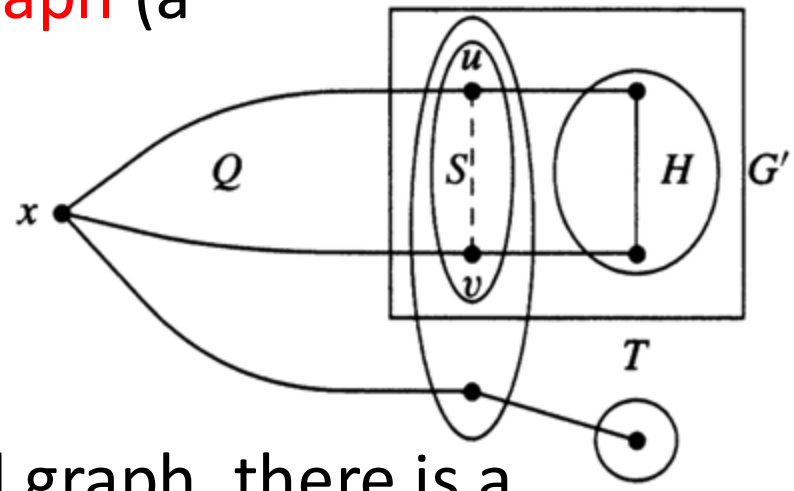



- Exercise (Ex 5.3.19, W) There exists some graph without simplicial elimination ordering but has a nice factorization form for chromatic polynomial
  - The existence of simplicial elimination ordering is a sufficient condition for the chromatic polynomial having all real roots, but not necessary

# Chordal graphs




- A chord of a cycle $C$ is an edge not in $C$ whose endpoints lie in $C$
- A chordless cycle in $G$ is a cycle of length at least $4$ that has no chord
- Theorem (5.3.17, W; Dirac 1961) A simple graph has a simplicial elimination ordering ⇔ it is a chordal graph (a simple graph without chordless cycle)
- TONCAS!
- Further $\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$ does not have a degree-1 decomposition

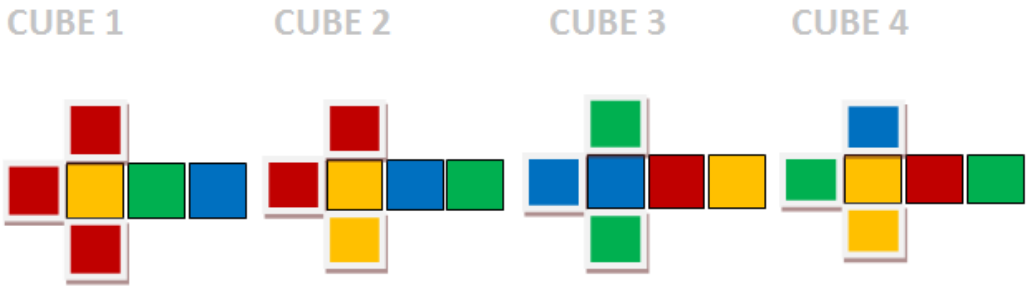


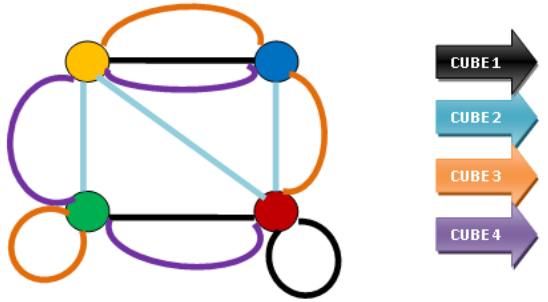

- Lemma (5.3.16, W) For every vertex $x$ in a chordal graph, there is a simplicial vertex of $G$ among the vertices farthest from $x$

# Proof Using Coloring

# Example -- Instant Insanity 四色方柱问题 (1.2, L)

- Problem make a stack of these cubes so that all four colors appear on each of the four sides of the stack
- An edge indicates that the two adjacent colors occur on opposite faces of the cube
- Problem necessary to find two subgraphs s.t.
  - are regular of degree 2
  - four edges, one from each cube
  - no edge in common for the two subgraphs



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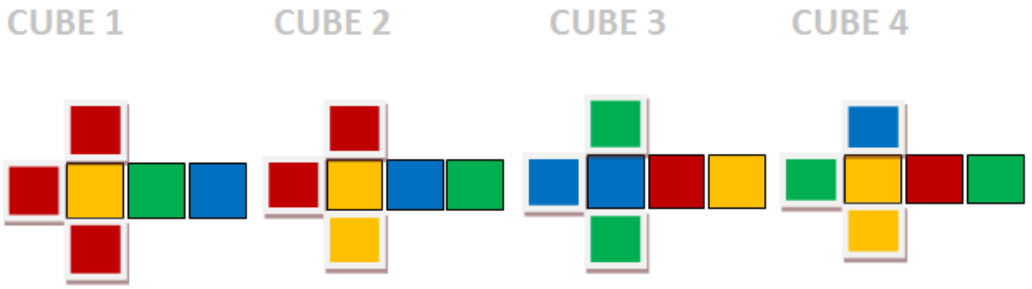
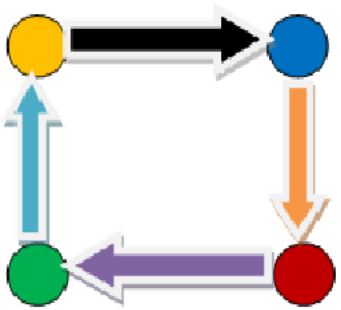
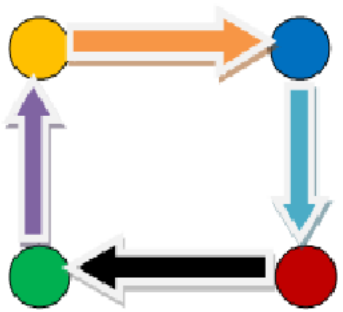
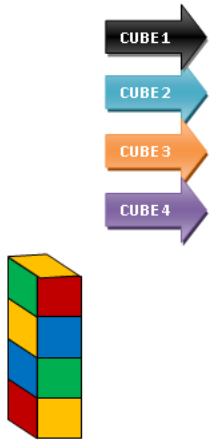


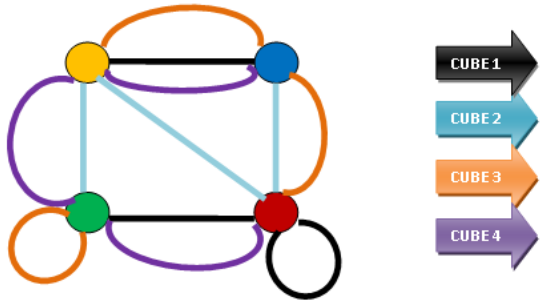



FRONT -> BACK



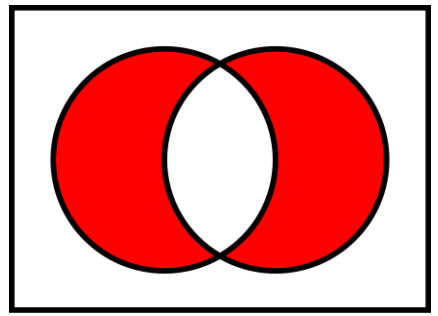
LEFT -> RIGHT



# An example about sets (1E, L)

- Let $A_1, \dots, A_n$ be $n$ distinct subsets of the $n$-set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \le i \le n$, are all distinct
- Proof Consider a graph with vertices $A_1, \dots, A_n$.
  - An edge of `color' $x$ between $A_i$ and $A_j$ iff $A_i \Delta A_j = \{x\}$
  - Then the problem is equivalent to find $y$ s.t. no color $y$
  - Notice that a cycle in this graph must have even length and each color appears even times
  - Then we can remove an edge if there is an edge with same color
  - Thus the number of colors remain the same and no cycle exists
  - By tree property, the number of edges is at most $n - 1$



# Summary

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- Coloring, proper coloring, chromatic #
- Brooks's theorem
- Chromatic # vs. clique/independence #
- Turán graph
- Color-critical, w/ vertex/edge-connectivity
- Chromatic number $4$ has a $K_4$-subdivision
- Chromatic polynomials, chromatic recurrence, path/trees/cycles, properties
- Simplicial elimination ordering, chordal graph, TONCAS
- Examples of proof with coloring

**Questions?**