Lecture 8: Planarity

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https://shuaili8.github.io/Teaching/CS3330/index.html

Motivation

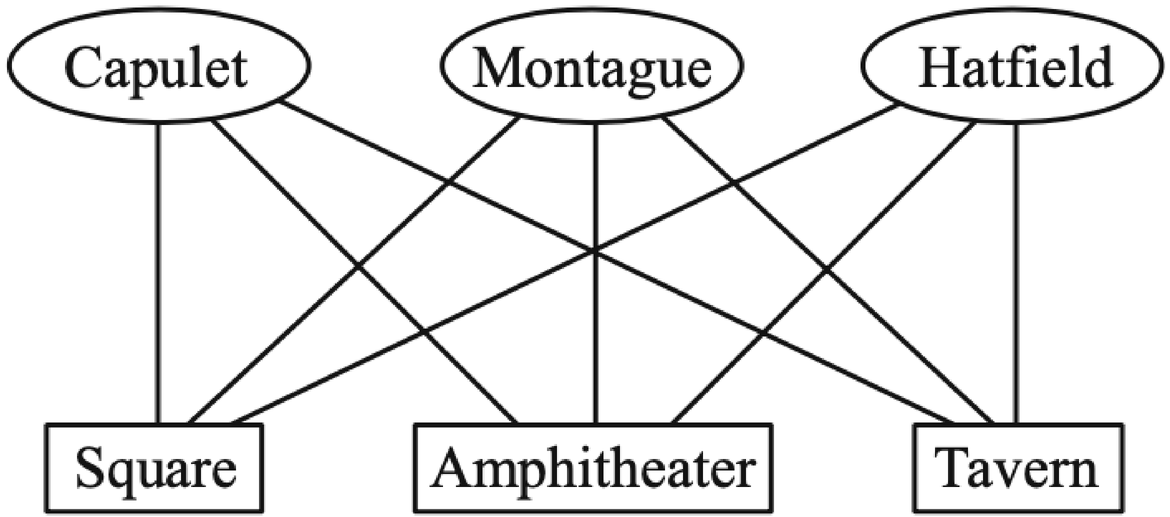


FIGURE 1.72. Original routes.

Definition and examples

- A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If G has no such representation, G is called nonplanar
- A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G

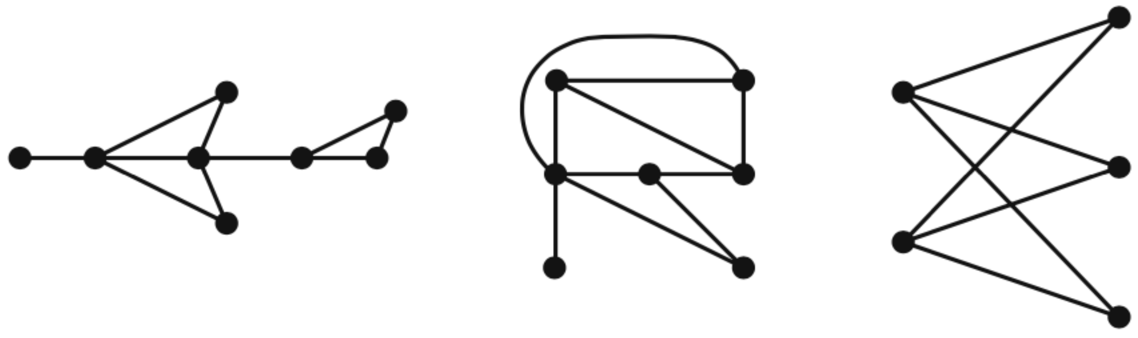
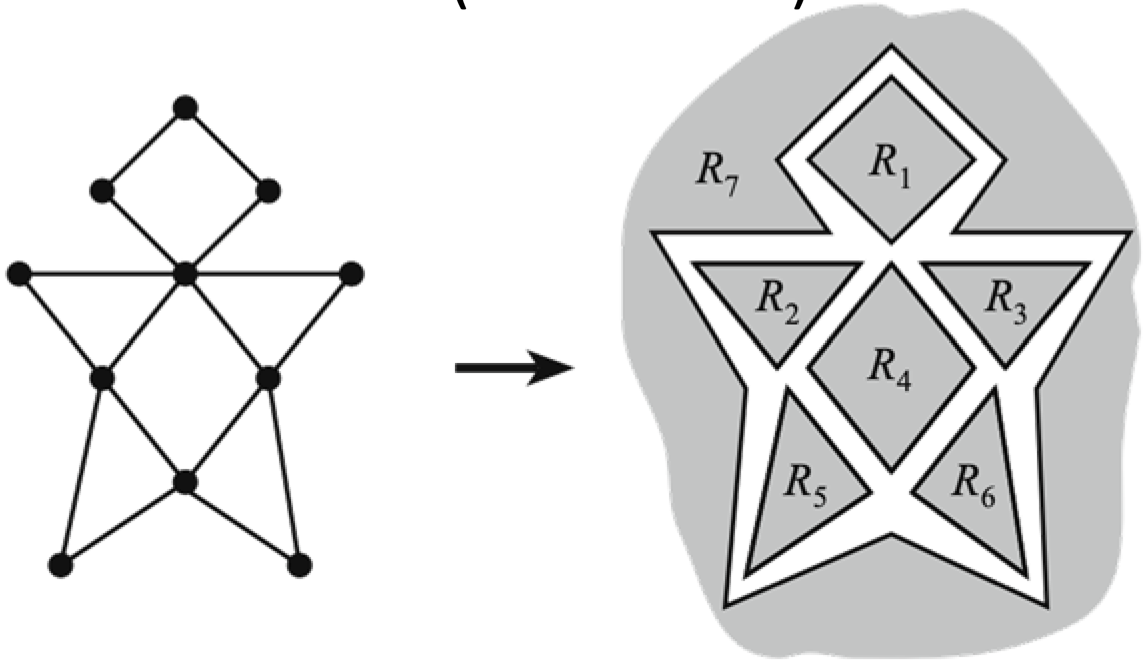


FIGURE 1.73. Examples of planar graphs.

Face

- Given a planar representation of a graph G, a face is a maximal region (polygonal open set) of the plane in which any two points can be joined by a curve that does not intersect any part of G
- The face R_7 is called the outer (or exterior) face



Face - properties

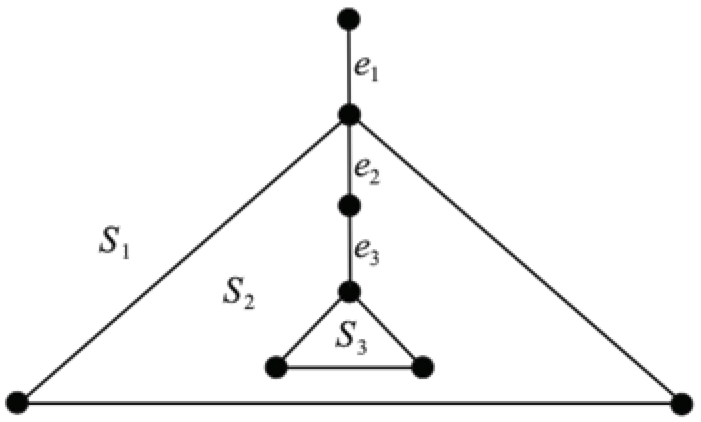


FIGURE 1.76. Edges e_1, e_2, and e_3 touch one face only.

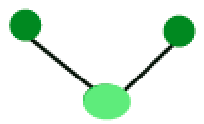
- An edge can come into contact with either one or two faces
- Example:
  - Edge e_1 is only in contact with one face S_1
  - Edge e_2, e_3 are only in contact with S_2
  - Each of other edges is in contact with two faces
- An edge e bounds a face F if e comes into contact with F and with a face different from F
- The bounded degree $b(F)$ is the number of edges that bound the face
  - Example: $b(S_1) = b(S_3) = 3, b(S_2) = 6$

Face - properties 2

- The length of a face in a plane graph G is the total length of the closed walk(s) in G bounding the face
- Proposition (6.1.13, W) If $l(F)$ denotes the length of face F in a plane graph G, then $2|E(G)| = \sum l(F_i)$
- Theorem (Restricted Jordan Curve Theorem) A simple closed polygonal curve C consisting of finitely many segments partitions the plane into exactly two faces, each having C as boundary

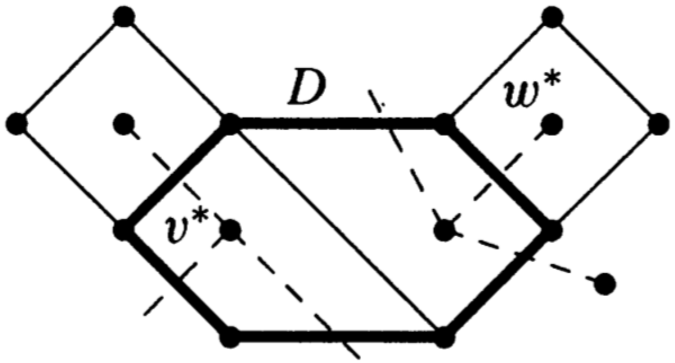
Bond

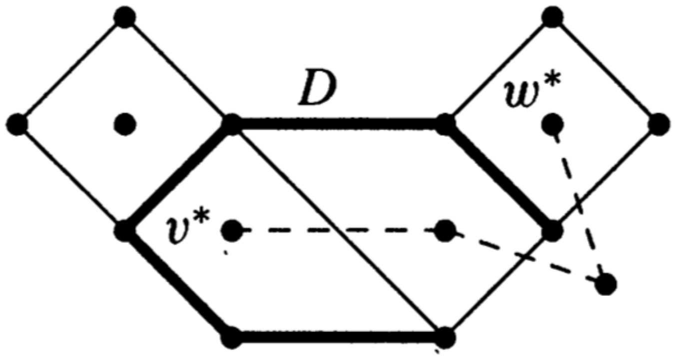
- An edge cut may contain another edge cut
- Example: $K_{1,2}$ or star graphs
- A bond is a minimal nonempty edge cut
- Proposition (4.1.15, W) If G is a connected graph, then an edge cut F is a bond $\Leftrightarrow$ $G - F$ has exactly two components



Dual graph

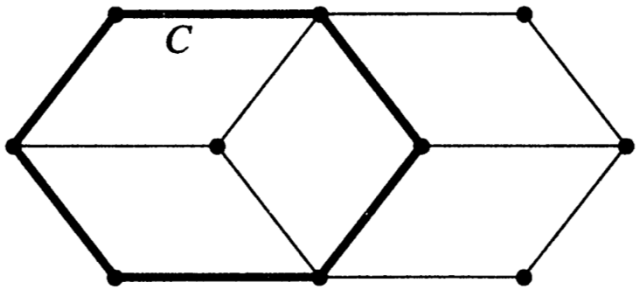
- The dual graph G^* of a plane graph G is a plane graph whose vertices are faces of G and edges are those contacting two faces
- Theorem (6.1.14, W) Edges in a plane graph G form a cycle in $G \Leftrightarrow$ the corresponding dual edges form a bond in G^*





Dual graph of bipartite graph

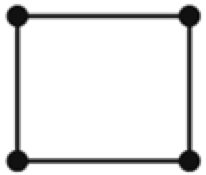
- Theorem (6.1.16, W) TFAE for a plane graph G
  - (a) G is bipartite
  - (b) Every face of G has even length
  - (c) The dual graph G^* is Eulerian



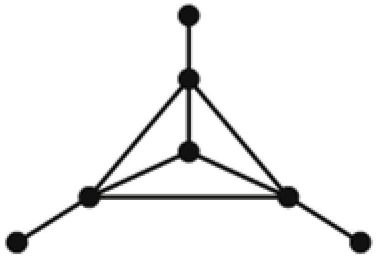
Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite $\Leftrightarrow$ it contains no odd cycle

The relationship between numbers of vertices, edges and faces

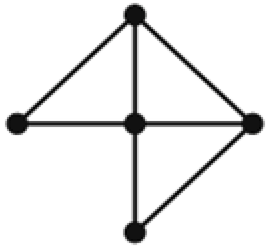
- The number of vertices n
- The number of edges m
- The number of faces f



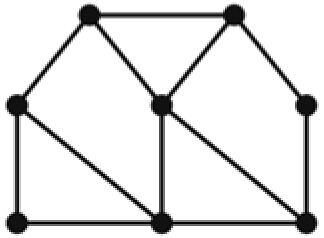
$n = 4$
$m = 4$
$f = 2$



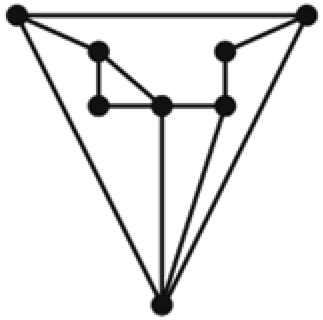
$n = 7$
$m = 9$
$f = 4$



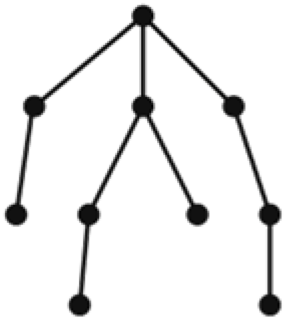
$n = 5$
$m = 7$
$f = 4$



$n = 8$
$m = 12$
$f = 6$



$n = 8$
$m = 12$
$f = 6$



$n = 10$
$m = 9$
$f = 1$

Euler's formula

- Theorem (1.31, H; 6.1.21, W; Euler 1758) If G is a connected planar graph with n vertices, m edges, and f faces, then
$$n - m + f = 2$$
  - Need Lemma: (Ex4, S1.5.1, H) Every tree is planar
- (Ex6, S1.5.2, H) Let G be a planar graph with k components. Then
$$n - m + f = k + 1$$

$K_{3,3}$ is nonplanar

- Theorem (1.32, H) $K_{3,3}$ is nonplanar

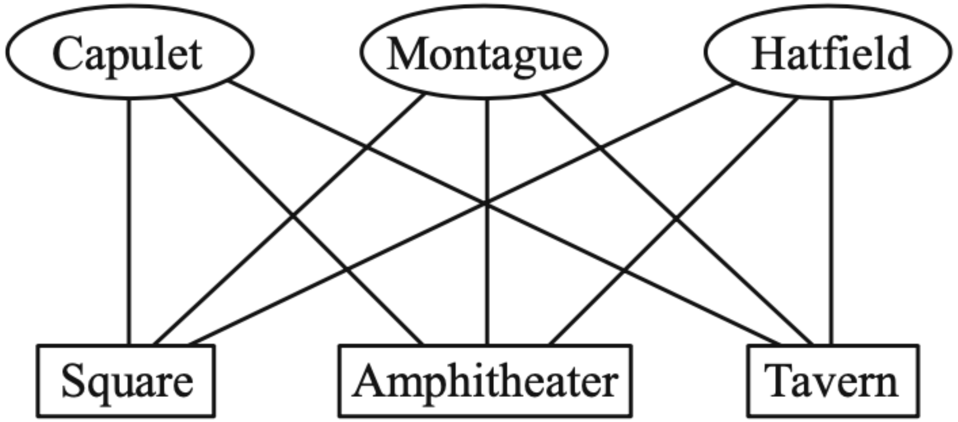


FIGURE 1.72. Original routes.

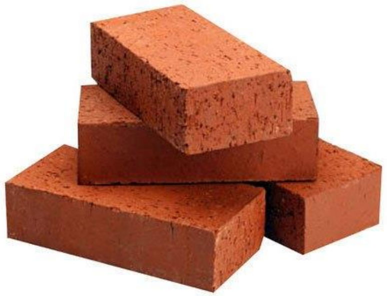
Upper bound for m

- Theorem (1.33, H; 6.1.23, W) If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds, then every face is bounded by 3 edges. In this case, G is maximal
- (Ex4, S1.5.2, H) Let G be a connected, planar, K_3-free graph of order $n \geq 3$. Then G has no more than $2n - 4$ edges
- Corollary (1.34, H) K_5 is nonplanar
- Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$
- (Ex5, S1.5.2, H) If G is bipartite planar graph, then $\delta(G) < 4$

Polyhedra

(Convex) Polyhedra 多面体

- A polyhedron is a solid that is bounded by flat surfaces



Polyhedra are planar

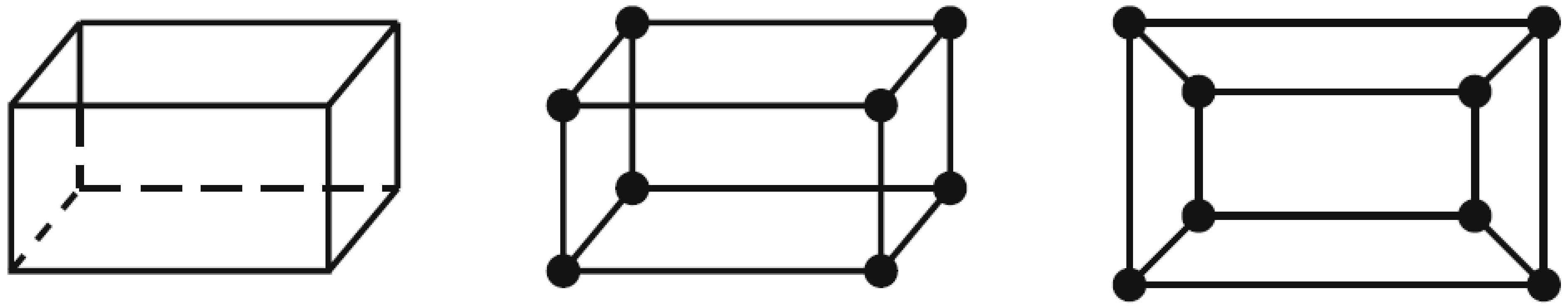
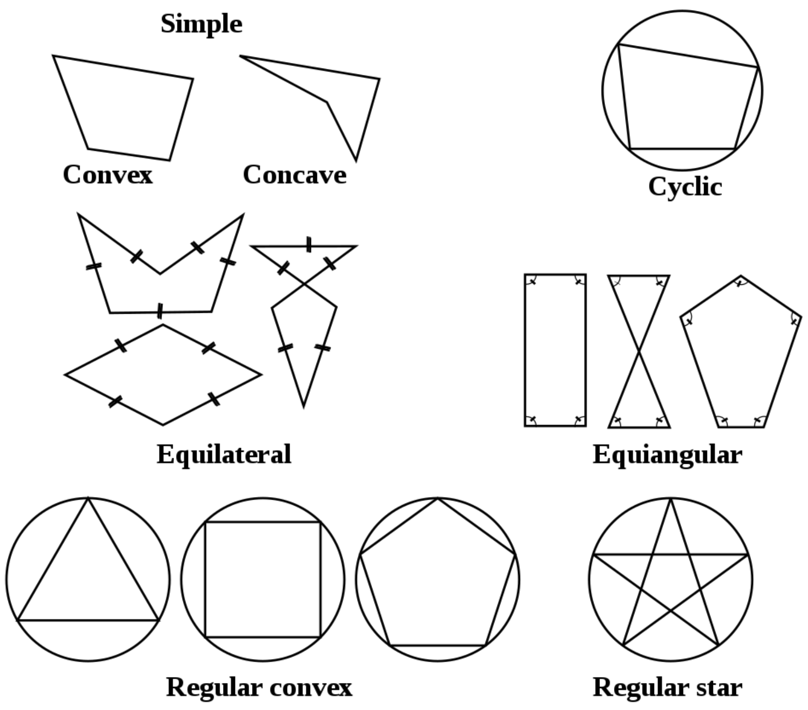


FIGURE 1.81. A polyhedron and its graph.

Properties

- Theorem (1.36, H) If a polyhedron has n vertices, m edges, and f faces, then
$$n - m + f = 2$$
- Given a polyhedron P, define
$$\rho(P) = \min\{l(F): F \text{ is a face of } P\}$$
- Theorem (1.37, H) For all polyhedron P, $3 \leq \rho(P) \leq 5$

Regular polyhedron 正多面体



- A regular polygon is one that is equilateral and equiangular
  正多边形(cycle)，等边、等角
- A polyhedron is regular if its faces are mutually congruent, regular polygons and if the number of faces meeting at a vertex is the same for every vertex
  正多面体
  面是相互全等的、正多边形、点的度数相等

Regular polyhedron 正多面体

- Theorem (1.38, H; 6.1.28, W) There are exactly five regular polyhedral
- 正四面体
- 立方体（正六面体）
- 正八面体
- 正十二面体
- 正二十面体

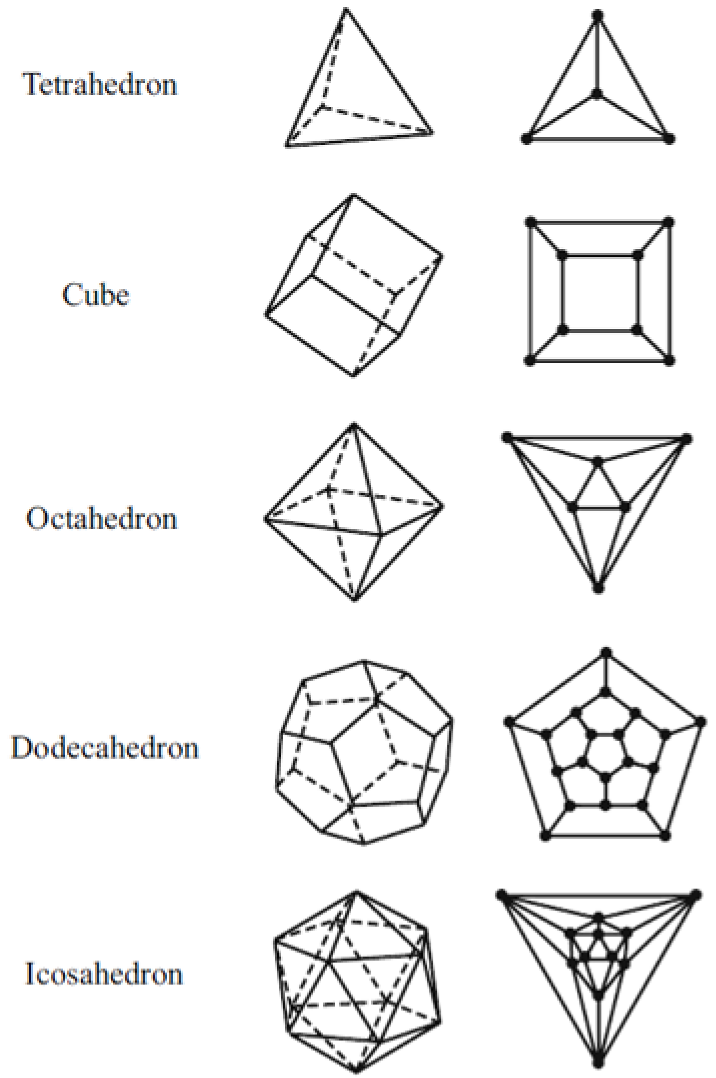


FIGURE 1.82. The five regular polyhedra and their graphical representations.

Kuratowski’s Theorem

Kuratowski's Theorem

- Theorem (1.39, H; Ex1, S1.5.4, H) A graph G is planar $\Leftrightarrow$ every subdivision of G is planar

- Theorem (1.40, H; Kuratowski 1930) A graph is planar $\Leftrightarrow$ it contains no subdivision of $K_{3,3}$ or K_5

The Four Color Problem

The Four Color Problem

- Q: Is it true that the countries on any given map can be colored with four or fewer colors in such a way that adjacent countries are colored differently?
- Theorem (Four Color Theorem) Every planar graph is 4-colorable
- Theorem (Five Color Theorem) (1.47, H; 6.3.1, W) Every planar graph is 5-colorable

Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$

- Exercise (Ex5, S1.6.3, H) Where does the proof go wrong for four colors?

Summary

- Planarity
- Dual graph
- Euler’s formula
- There are exactly five regular polyhedral
- Kuratowski’s Theorem
- Four/Five Color Theorem

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Questions?