# Lecture 9: Ramsey Theory

Shuai Li

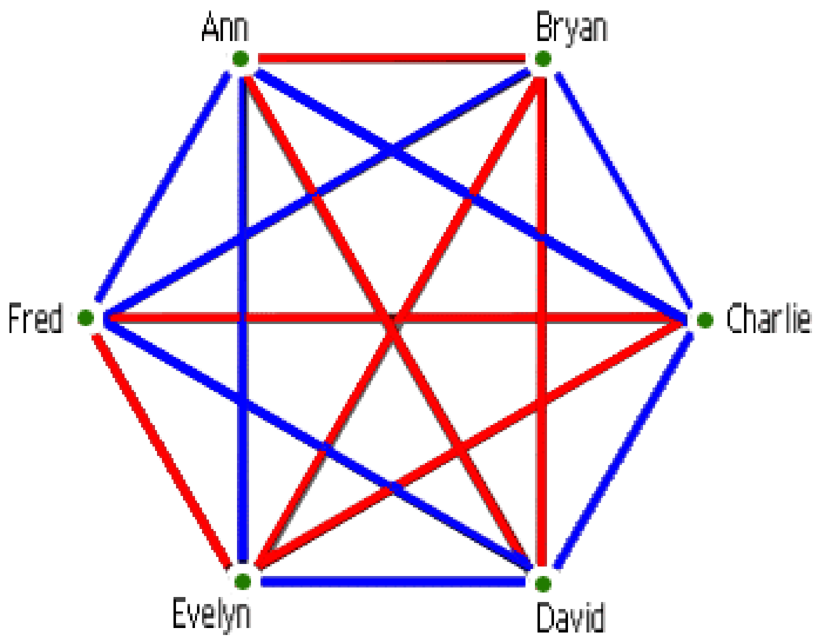
John Hopcroft Center, Shanghai Jiao Tong University

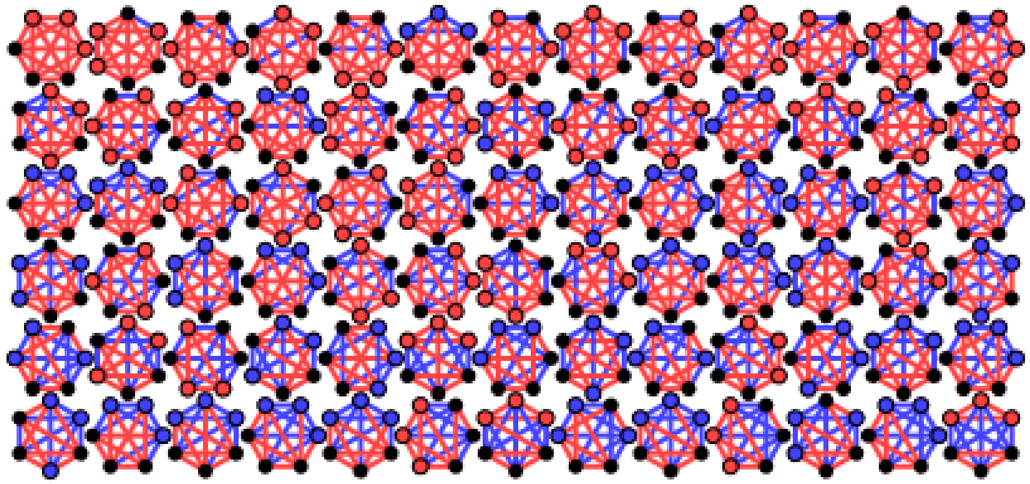
https://shuaili8.github.io

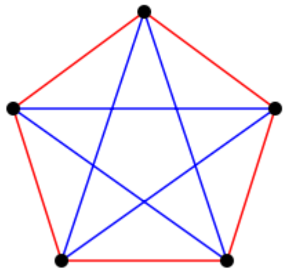
https://shuaili8.github.io/Teaching/CS3330/index.html

# The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?



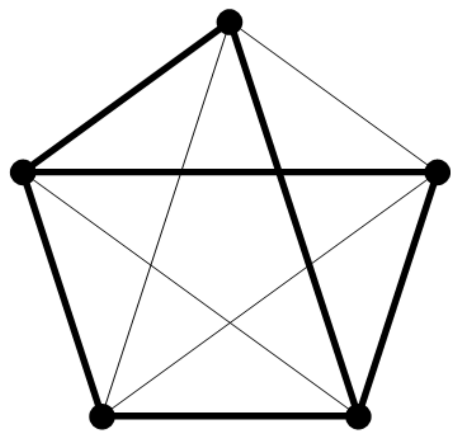


R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

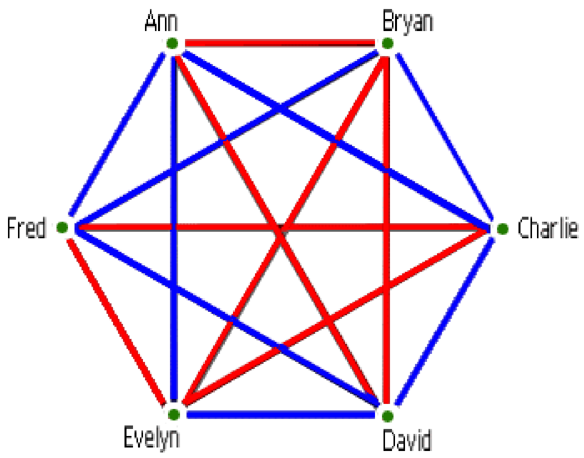
https://plus.maths.org/content/friends-and-strangers
Wikipedia

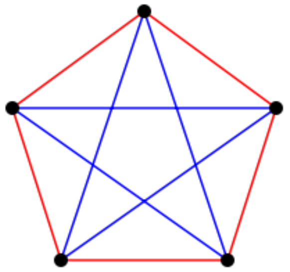
# (classical) Ramsey number

- A 2-coloring of the edges of a graph $G$ is any assignment of one of two colors of each of the edges of $G$
- Let $p$ and $q$ be positive integers. The (classical) Ramsey number associated with these integers, denoted by $R(p, q)$, is defined to be the smallest integer $n$ such that every 2-coloring of the edges of $K_n$ either contains a red $K_p$ or a blue $K_q$ as a subgraph
- It is a typical problem of extremal graph theory



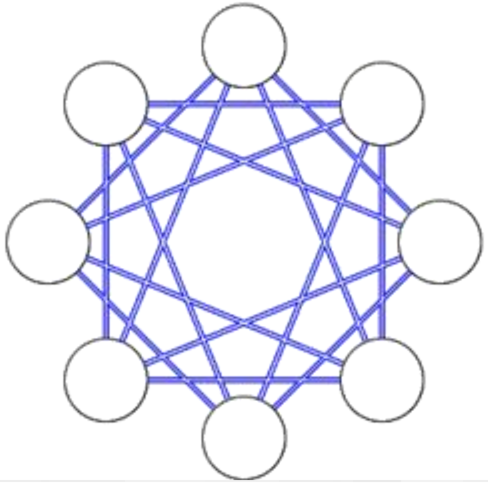
# Examples

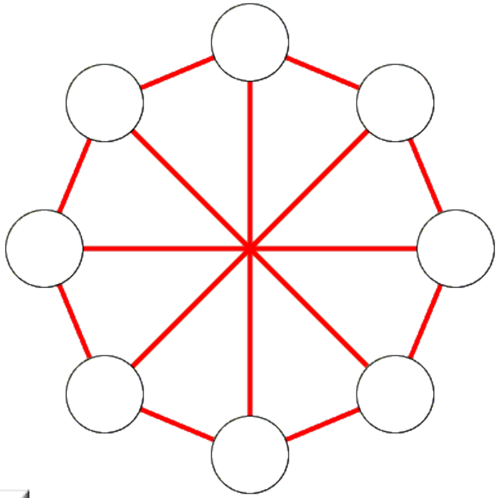


- $R(1,3) = 1$
- (Ex2, S1.8.1, H) $R(1,k) = 1$
- $R(2,4) = 4$
- (Ex3, S1.8.1, H) $R(2,k) = k$
- Theorem (1.61, H; 8.3.1, 8.3.9, W) $R(3,3) = 6$

# Examples (cont.)





- Theorem (1.62, H; 8.3.10, W) $R(3,4) = 9$

Theorem A finite graph $G$ has an even number of vertices with odd degree

- (Ex4, S1.8.1, H) $R(p,q) = R(q,p)$

Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

| r \ s | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
|---|---|---|---|---|---|---|---|---|---|---|
| 1 | **1** | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 2 | | **2** | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 |
| 3 | | | **6** | 9 | 14 | 18 | 23 | 28 | 36 | 40–42 |
| 4 | | | | **18** | 25[10] | 36–41 | 49–61 | 59[14]–84 | 73–115 | 92–149 |
| 5 | | | | | **43–48** | 58–87 | 80–143 | 101–216 | 133–316 | 149[14]–442 |
| 6 | | | | | | **102–165** | 115[14]–298 | 134[14]–495 | 183–780 | 204–1171 |
| 7 | | | | | | | **205–540** | 217–1031 | 252–1713 | 292–2826 |
| 8 | | | | | | | | **282–1870** | 329–3583 | 343–6090 |
| 9 | | | | | | | | | **565–6588** | 581–12677 |
| 10 | | | | | | | | | | **798–23556** |

# Bounds on Ramsey numbers

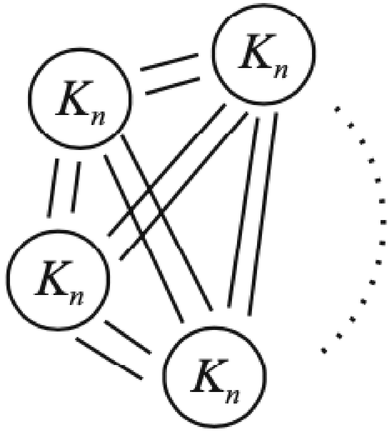
- Theorem (1.64, H; 2.28, H; 8.3.11, W) If $q \geq 2, q \geq 2$, then
$$R(p,q) \leq R(p-1,q) + R(p,q-1)$$
Furthermore, if both terms on the RHS are even, then the inequality is strict

> Theorem A finite graph $G$ has an even number of vertices with odd degree

- Theorem (1.63, H; 2.29, H) $R(p,q) \leq \binom{p+q-2}{p-1}$
- Theorem (1.65, H) For integer $q \geq 3$, $R(3,q) \leq \frac{q^2+3}{2}$
- Theorem (1.66, H; 8.3.12, W; Erdős and Szekeres 1935)
If $p \geq 3$, $R(p,p) > \lfloor 2^{p/2} \rfloor$

# Graph Ramsey Theory

- Given two graphs $G$ and $H$, define the graph Ramsey number $R(G, H)$ to be the smallest value of $n$ such that any 2-coloring of the edges of $K_n$ contains either a red copy of $G$ or a blue copy of $H$
  - The classical Ramsey number $R(p, q)$ would in this context be written as $R(K_p, K_q)$
- Theorem (1.67, H) If $G$ is a graph of order $p$ and $H$ is a graph of order $q$, then $R(G, H) \leq R(p, q)$
- Theorem (1.68, H) Suppose the order of the largest component of $H$ is denoted as $C(H)$.
  Then $R(G, H) \geq (\chi(G) - 1)(C(H) - 1) + 1$



# Graph Ramsey Theory (cont.)

- Theorem (1.69, H; 8.3.14, W) $R(T_m, K_n) = (m-1)(n-1)+1$

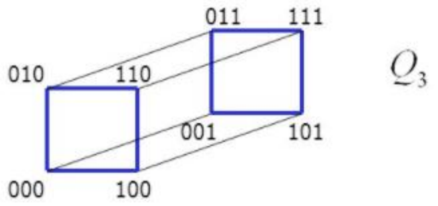
Theorem (1.45, H; Ex6, S1.6.2, H) For any graph $G$ of order $n$,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n+1-\alpha(G)$$

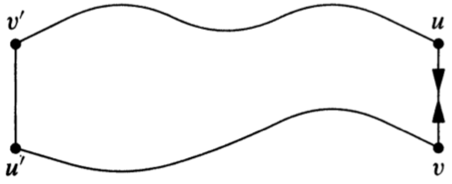
Proposition (5.2.13, W) Let $G$ be a $k$-critical graph
(a) For every $v \in V(G)$, there is a proper coloring such that $v$ has a unique color and other $k-1$ colors all appear on $N(v)$
$\Rightarrow \delta(G) \geq k-1$

Theorem (1.16, H) Let $T$ be a tree of order $k+1$ with $k$ edges. Let $G$ be a graph with $\delta(G) \geq k$. Then $G$ contains $T$ as a subgraph

# More on pigeonhole principle




- Proposition (8.3.1, W) Among 6 people, it is possible to find 3 mutual acquaintances or 3 mutual non-acquaintances
  - ⇔ For every simple graph with 6 vertices, there is a triangle in $G$ or in $\bar{G}$
- Theorem (8.3.2, W) If $T$ is a spanning tree of the $k$-dimensional cube $Q_k$, then there is an edge of $Q_k$ outside $T$ whose addition to $T$ creates a cycle of length at least $2k$




$T$ is a tree of order $n$
⇔ Any two vertices of $T$ are linked by a unique path in $T$

  - ⇒ Every spanning tree of $Q_k$ has diameter at least $2k-1$

# More on pigeonhole principle 2

- Theorem (8.3.3, W; Erdős–Szekeres 1935) Every list of $\geq n^2 + 1$ distinct numbers has a monotone sublist of length $\geq n + 1$
  - Generalization. $(r-1)(s-1)+1$
- Theorem (8.3.4, W; Graham-Kleitman 1973) In every labeling of $E(K_n)$ using distinct integers, there is a walk of length at least $n-1$ along which the labels strictly increase

# Summary

- Ramsey number
- Graph Ramsey Theory
- More on pigeonhole principle

**Shuai Li**

https://shuaili8.github.io

**Questions?**