上海交通大學
SHANGHAI JIAO TONG UNIVERSITY

上海交通大学
约翰·霍普克罗夫特
计算机科学中心
John Hopcroft Center for Computer Science

CS 3330: Combinatorics

Shuai Li & Biaoshuai Tao

John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://jhc.sjtu.edu.cn/~bstao/

https://shuaili8.github.io/Teaching/CS3330/index.html

Shuai Li (Week 1-8)

- Position
  - Assistant Professor at John Hopcroft Center since Sep 2019
- Education
  - PhD in Computer Science from the Chinese University of Hong Kong
  - Master in Math from the Chinese Academy of Sciences
  - Bachelor in Math from Zhejiang University
- Research interests
  - Reinforcement learning algorithms and analysis
  - Optimization, algorithms and analysis in machine learning

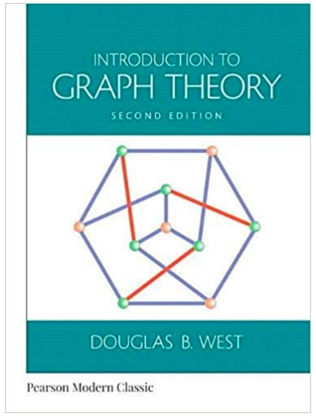
Biaoshuai Tao (Week 9-16)

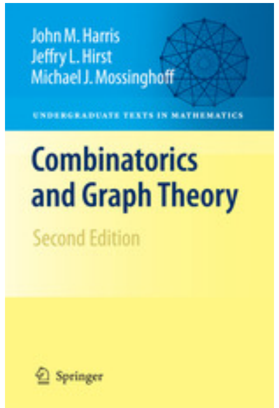
- Position
  - Assistant Professor at John Hopcroft Center since Nov 2020
- Education
  - PhD in Computer Science from the University of Michigan
  - Bachelor in Math from Nanyang Technological University
- Research interests
  - Theoretical Computer Science
  - Computational Economics

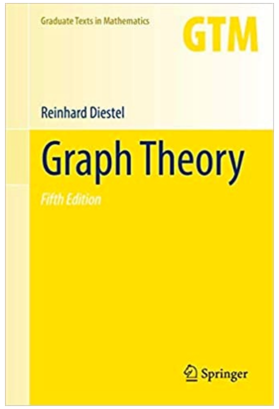
References

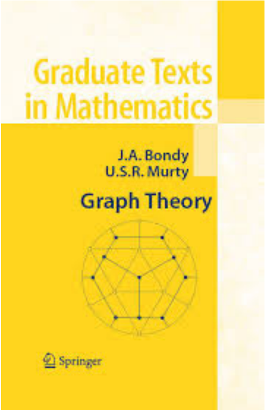
- References:
  - Introduction to Graph Theory, by Douglas West
  - Combinatorics and Graph Theory, by Harris, Hirst and Mossinghoff

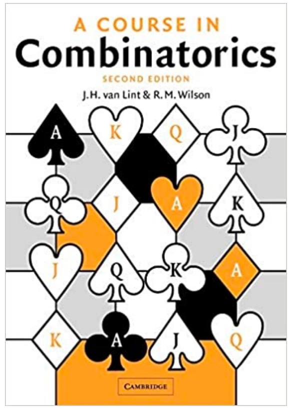
  - Graph Theory, Reinhard Diestel
  - Graph Theory, by Bondy and Murty
  - A Course in Combinatorics, J. H. Van Lint











Previous courses

- Discrete Mathematics
  - Basic concepts for graph theory
- Mathematical Foundations of Computer Science (CS 499?)
  - Basic notions and hand shaking lemma
  - Graph isomorphism and graph score
  - Applications of handshake lemma: Parity argument
  - The number of spanning trees
  - Isomorphism of trees
  - Random graphs

Goal

- Knowledge of the basic problems for graph theory
  - Bipartite graphs/Matching/Circuits/Coloring/…
- Knowledge of the important counting related results on graphs
- Familiar with the common proof techniques
- Awareness of the popular applications of graphs in many fields

Grading policy

- Attendance and participance: 5%
- Assignments: 35%
- Midterm exam: 15%
- Entry editing: 5%
- Reading report: 10%
- Final exam: 30%

Honor code

- Discussions are encouraged
- Independently write-up homework and project
- Same reports and homework will be reported

Teaching Assistant

- Ruofeng Yang (杨若峰) (Week 1-8)
  - Email: wanshuiyin@sjtu.edu.cn
  - 1st year PhD Student
  - Research interests on bandit algorithms and optimization
  - Office hour: Tue 7-9 PM
- Yichen Tao (陶浥尘) (Week 9-16)
  - Email: taoyc0904@sjtu.edu.cn
  - Senior undergraduate student
  - Research interests on economics & computation
  - Office hour: Mon 7-9 PM, Activity Room of No.1 Dormitory Building, East District
- Zilong Wang (王子龙) (Week 1-8)
  - Email: wangzilong@sjtu.edu.cn
  - Senior undergraduate student
  - Research interests on bandit and RL algorithms
  - Office hour: Wed 7-9 PM

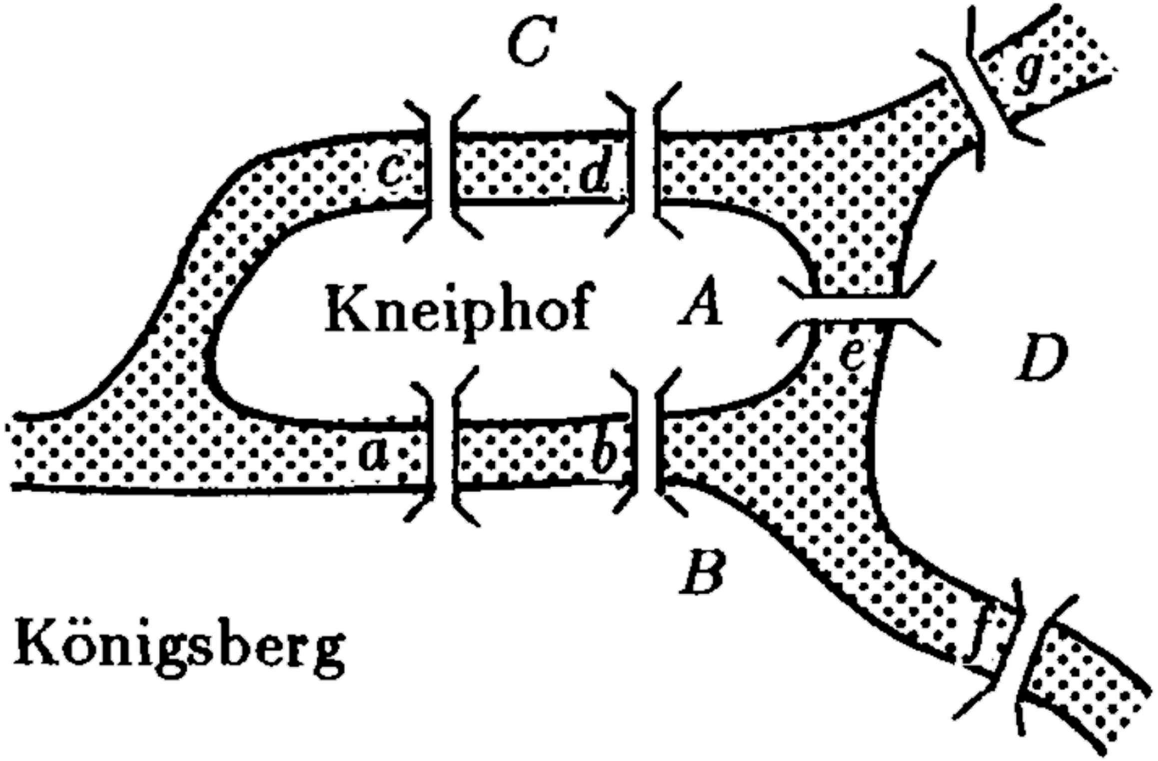
Course Outline

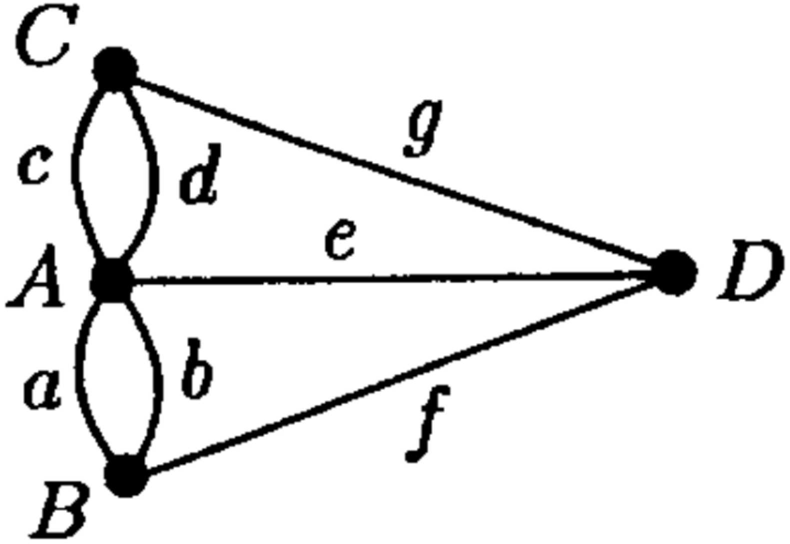
- Basics
  - Graphs, paths and cycles, connectivity, bipartite graphs
- Trees
- Matching
- Connectivity
- Planar Graphs
- Coloring
- Circuits

Introduction

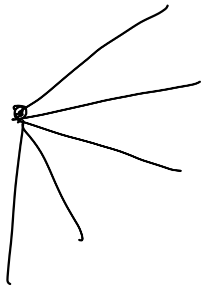
Seven bridges of Königsberg 七桥问题

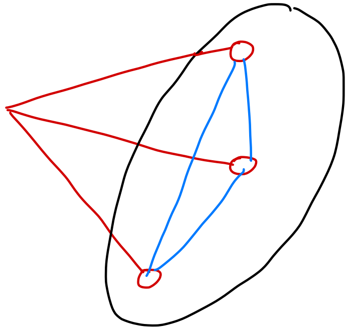
- Leonhard Euler 1736: Is it possible to make a walk through the city, returning to the starting point and crossing each bridge exactly once?



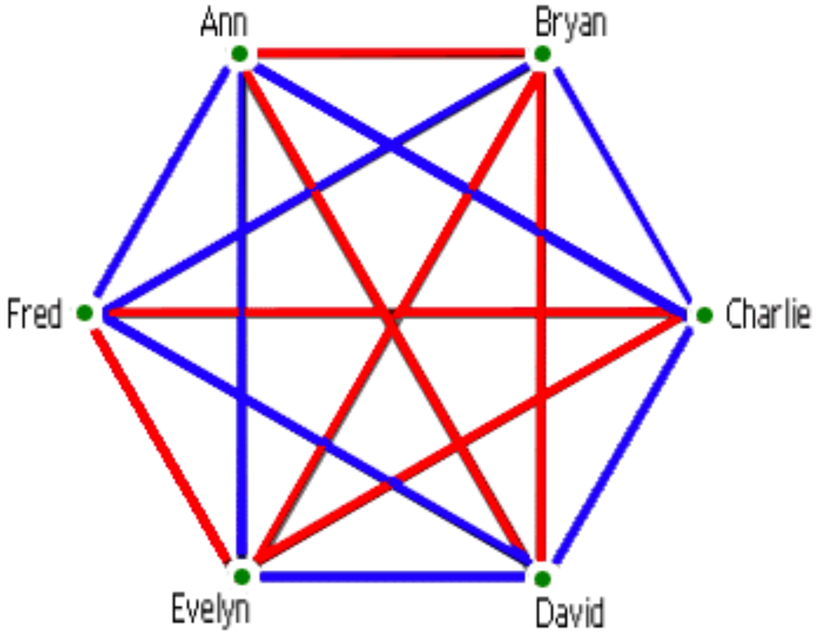


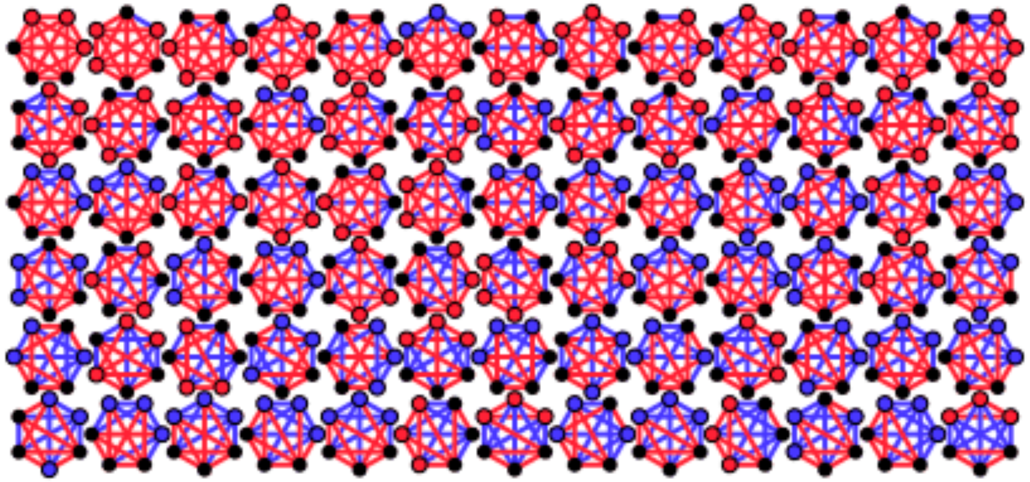
The friendship riddle

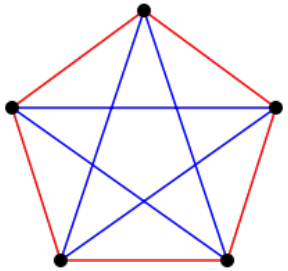




- Does every set of six people contain three mutual acquaintances or three mutual strangers?







R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

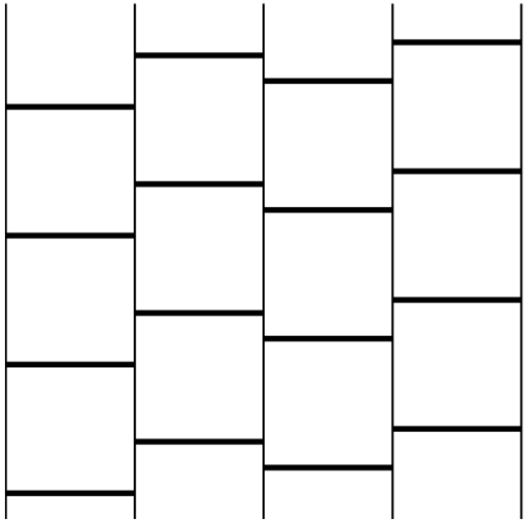
https://plus.maths.org/content/friends-and-strangers
Wikipedia

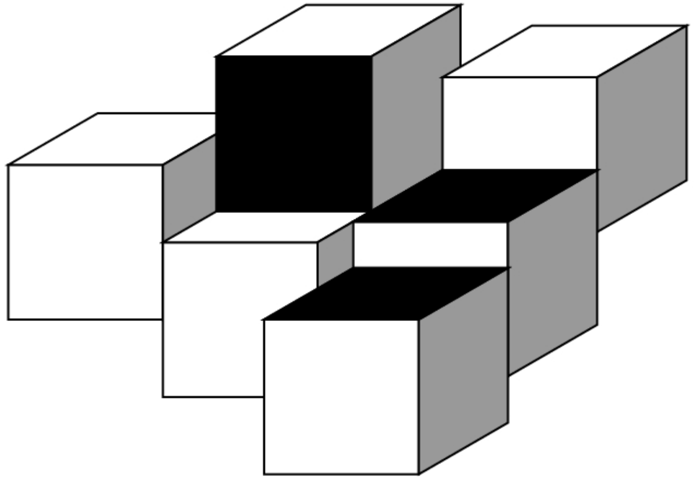
Examples of general combinatorics problems using graph theory

- Instant Insanity 四色方柱问题
  - make a stack of these cubes so that all four colors appear on each of the four sides of the stack

- A set problem
  - Let $A_1, \dots, A_n$ be n distinct subsets of the n-set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \le i \le n$, are all distinct

Keller’s conjecture



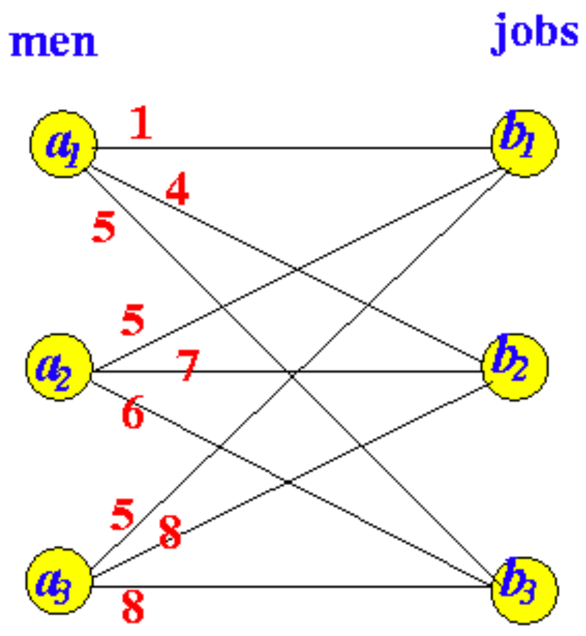


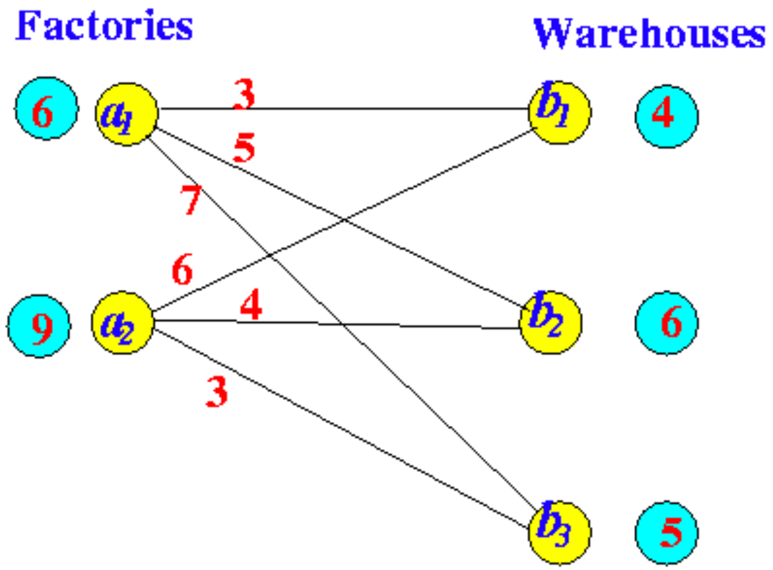
- In 1930, Keller conjectured that any tiling of n-dimensional space by translates of the unit cube must contain a pair of cubes that share a complete $(n-1)$-dimensional face

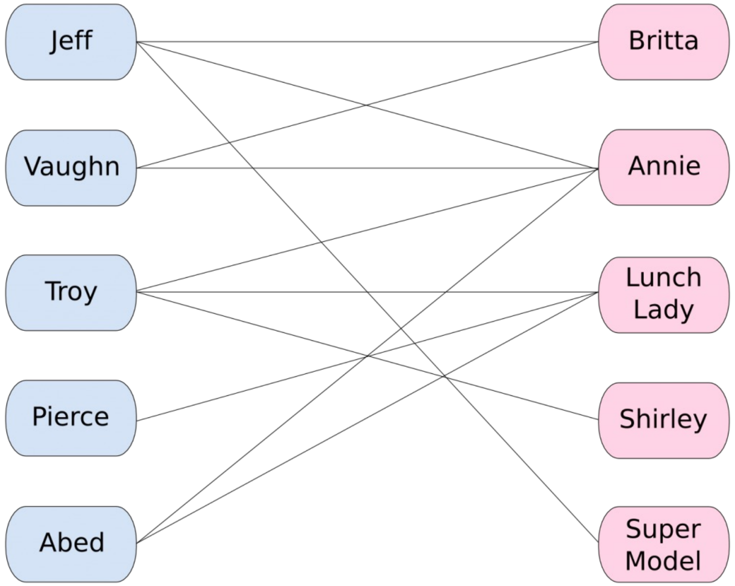
- Corrádi and Szabó transfer it into a graph theory problem
  - Constructing Keller graph
- The conjecture is solved by computer search recently (June 2020)

Reference: https://mp.weixin.qq.com/s/rTxW6MzSHuWUGC3vTIDbJw

Assignment problems

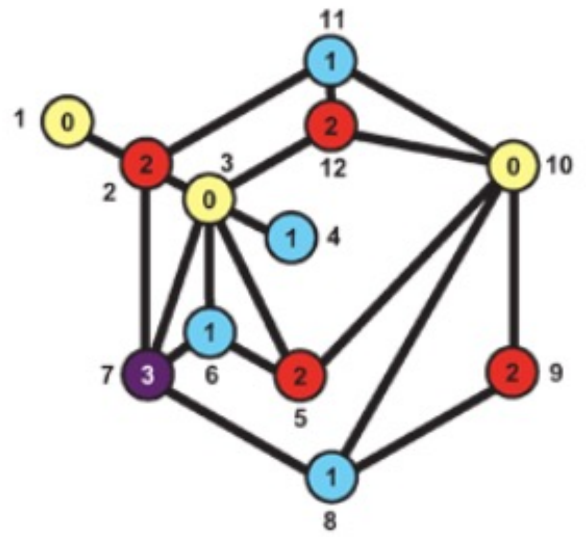






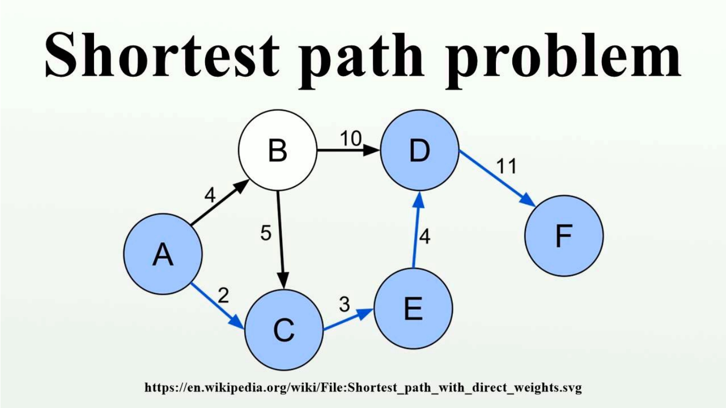
Scheduling and coloring

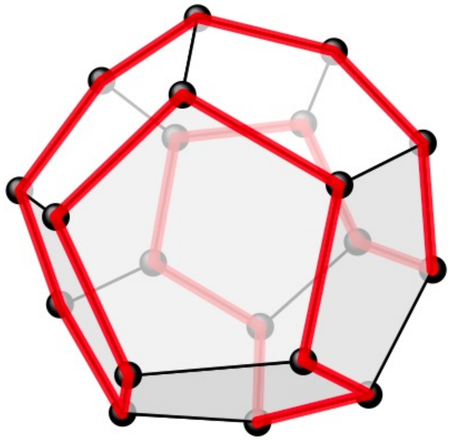
- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member

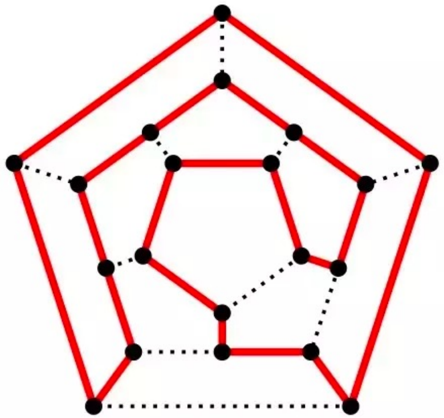


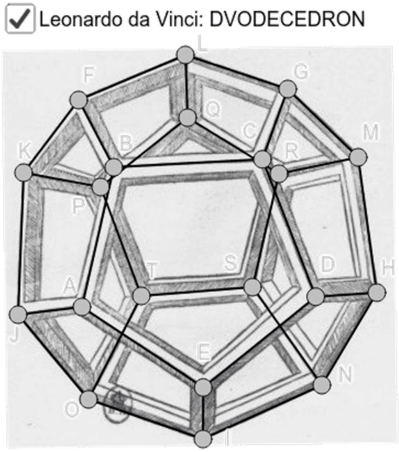
Routes in road networks

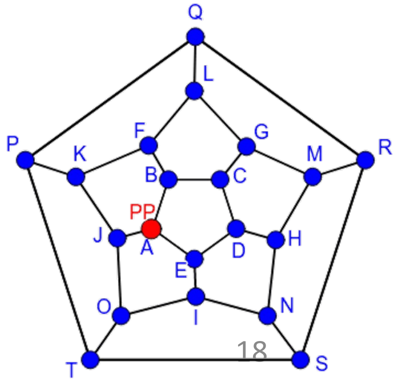
- How can we find the shortest route from A to F?
- If the vertices of the graph represent our house and other places to visit, then we may want to follow a route that visits every vertex exactly once, so as to visit everyone without overstaying our welcome
  - Hamilton circuit





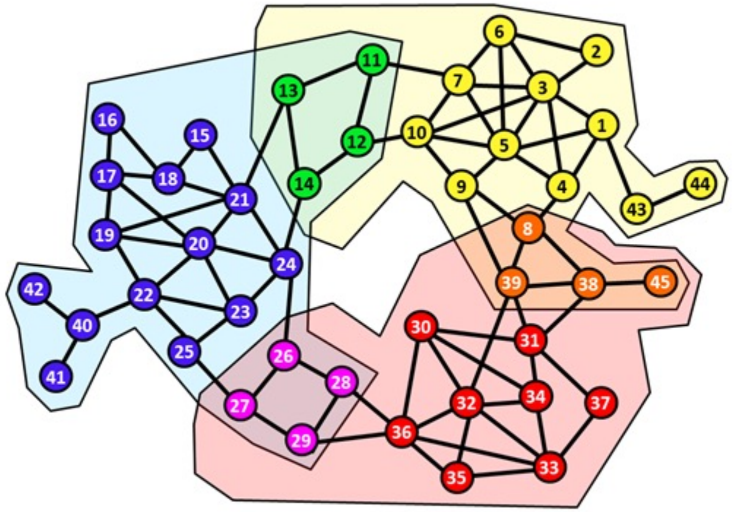


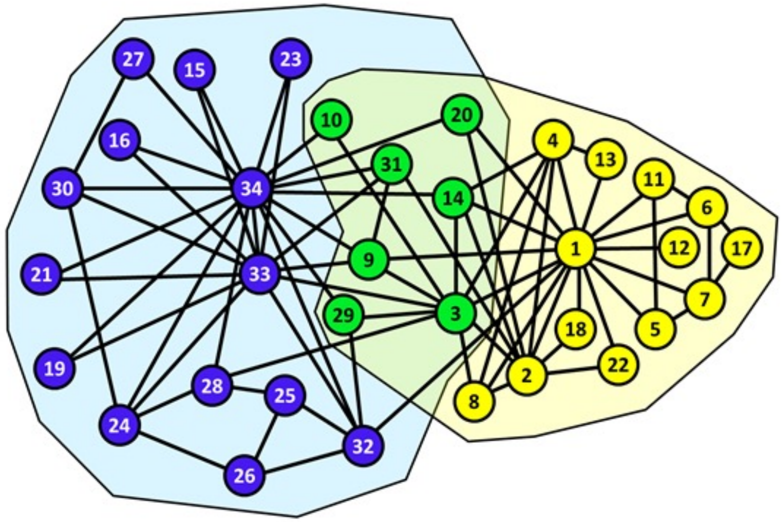




Social network

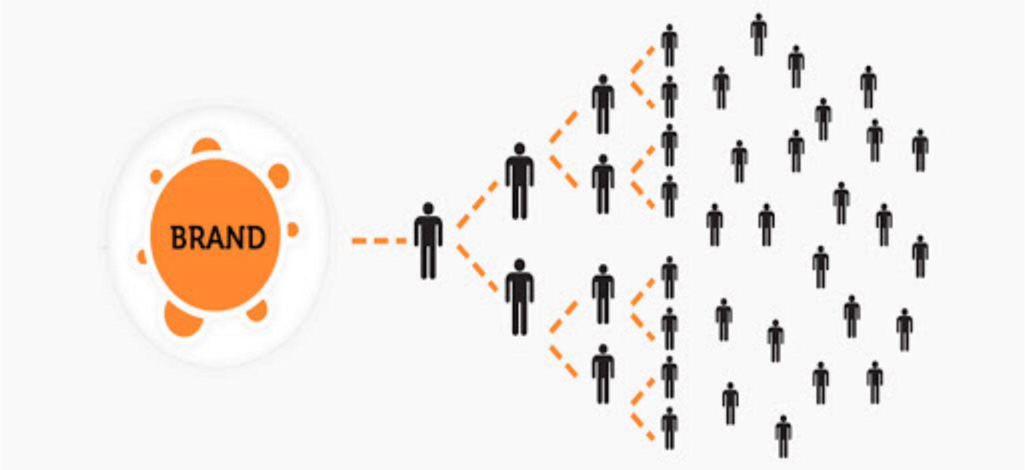
- Recommendation
- Clustering

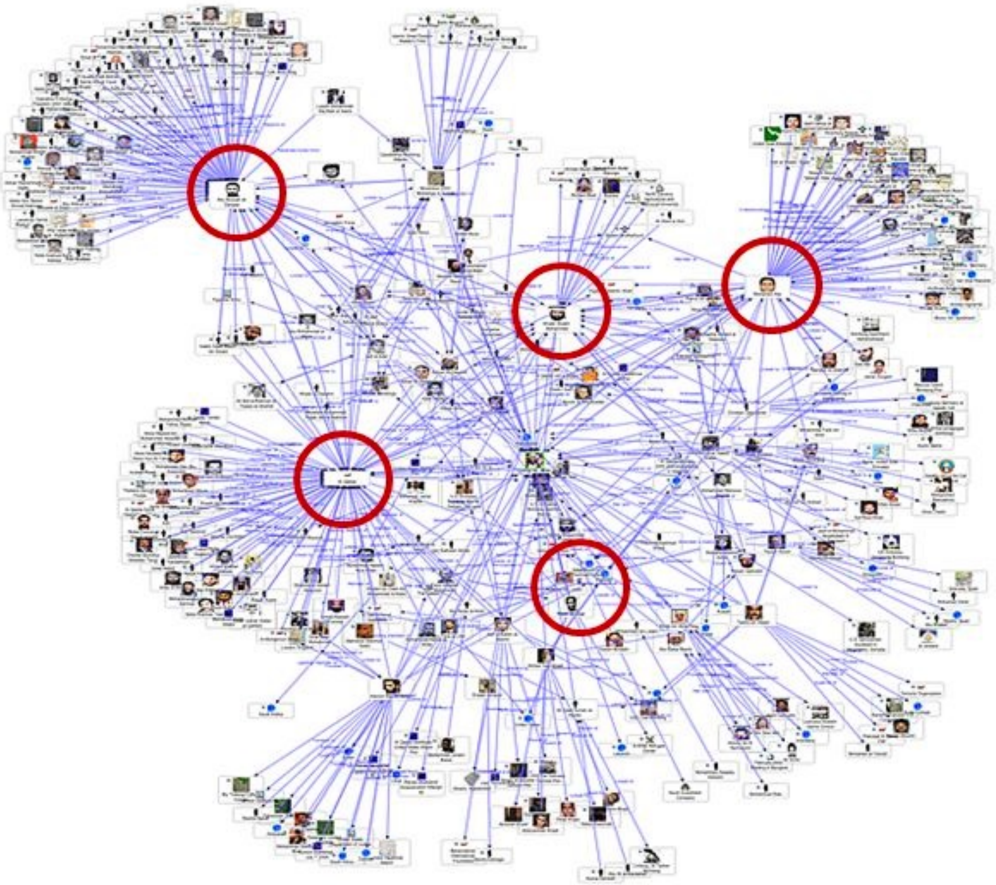




Influence maximization

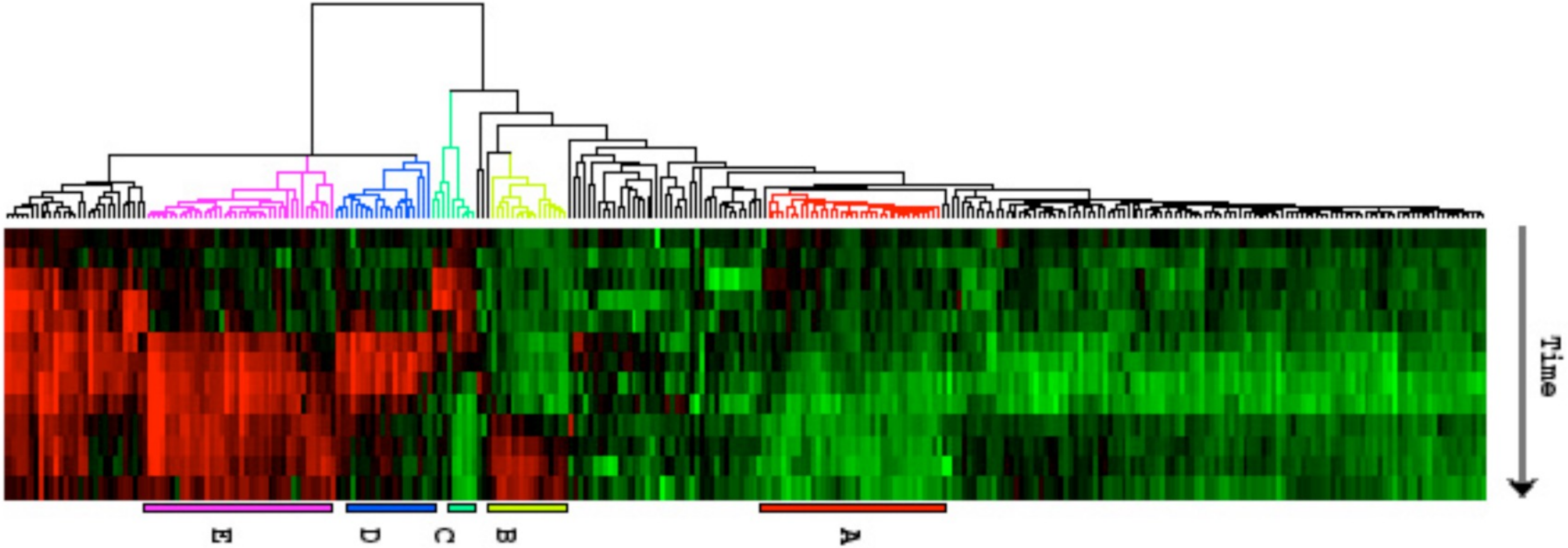
- Select the best seed set



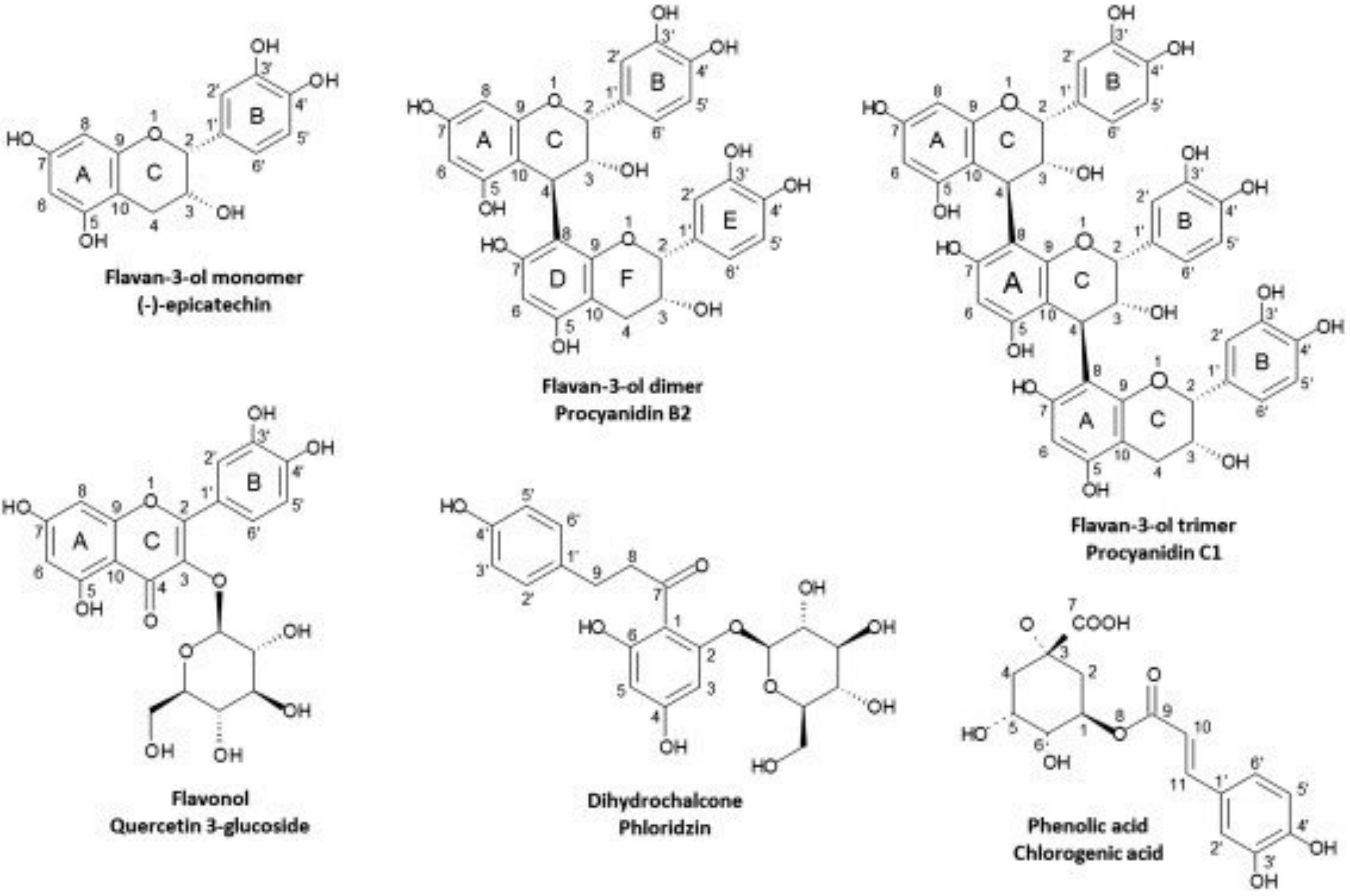


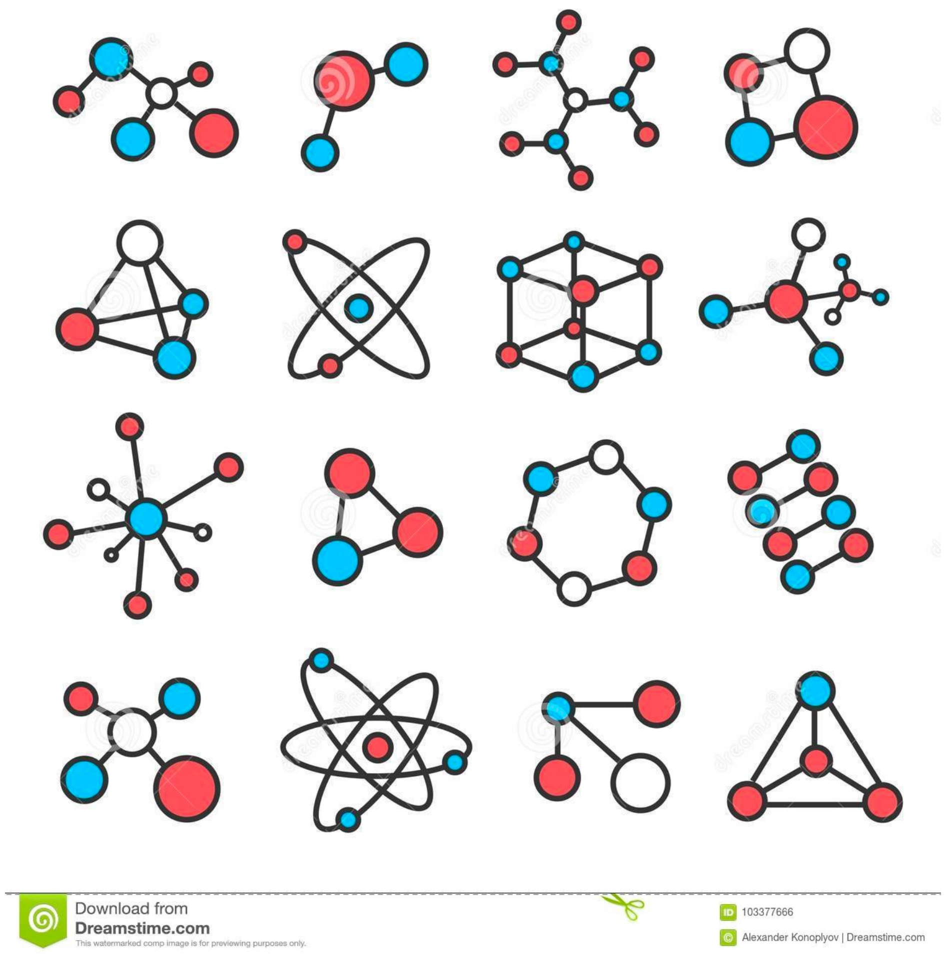
Gene structure

- Tree graph
  - Agglomerative clustering method

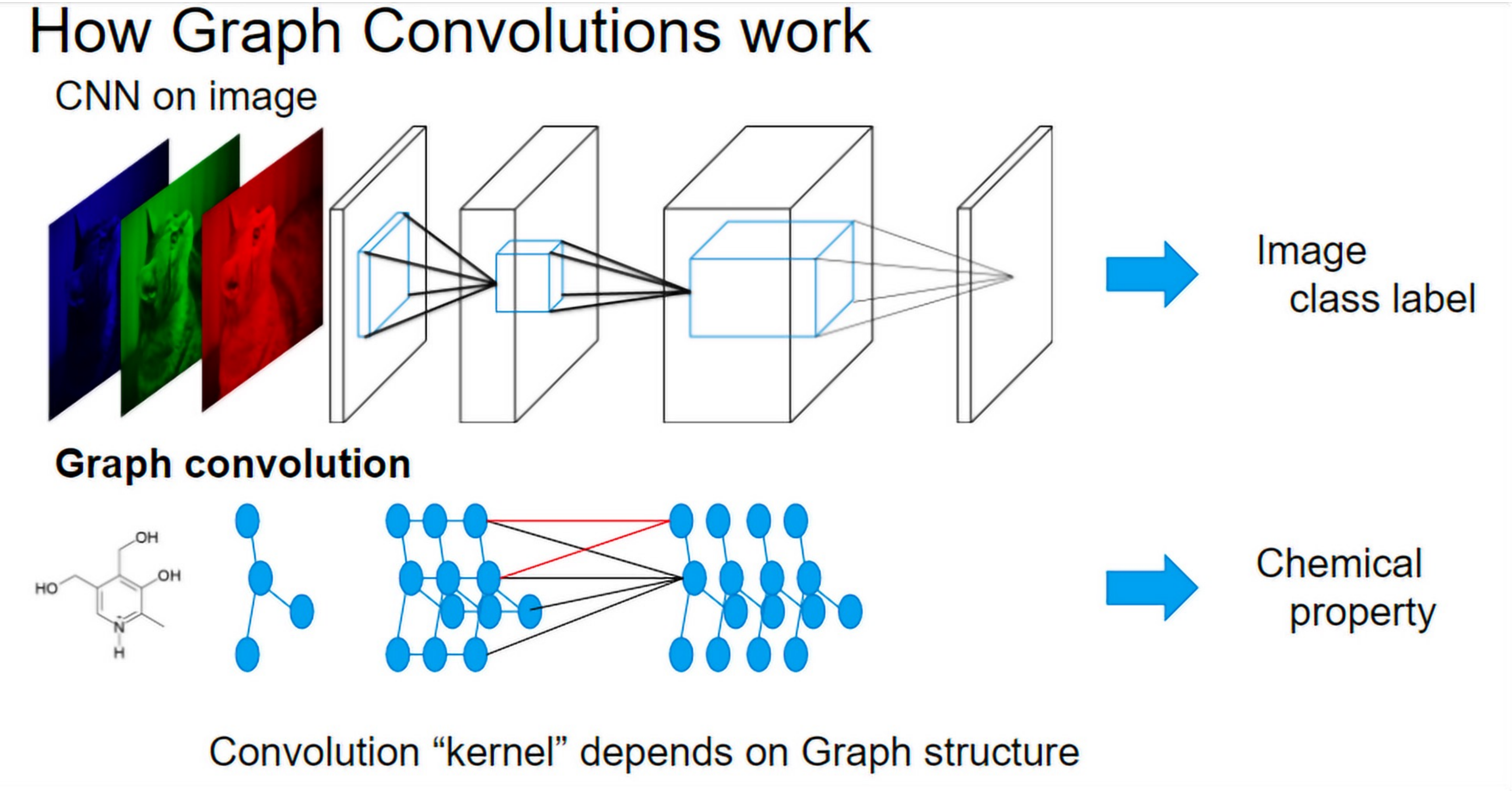


Molecular structure

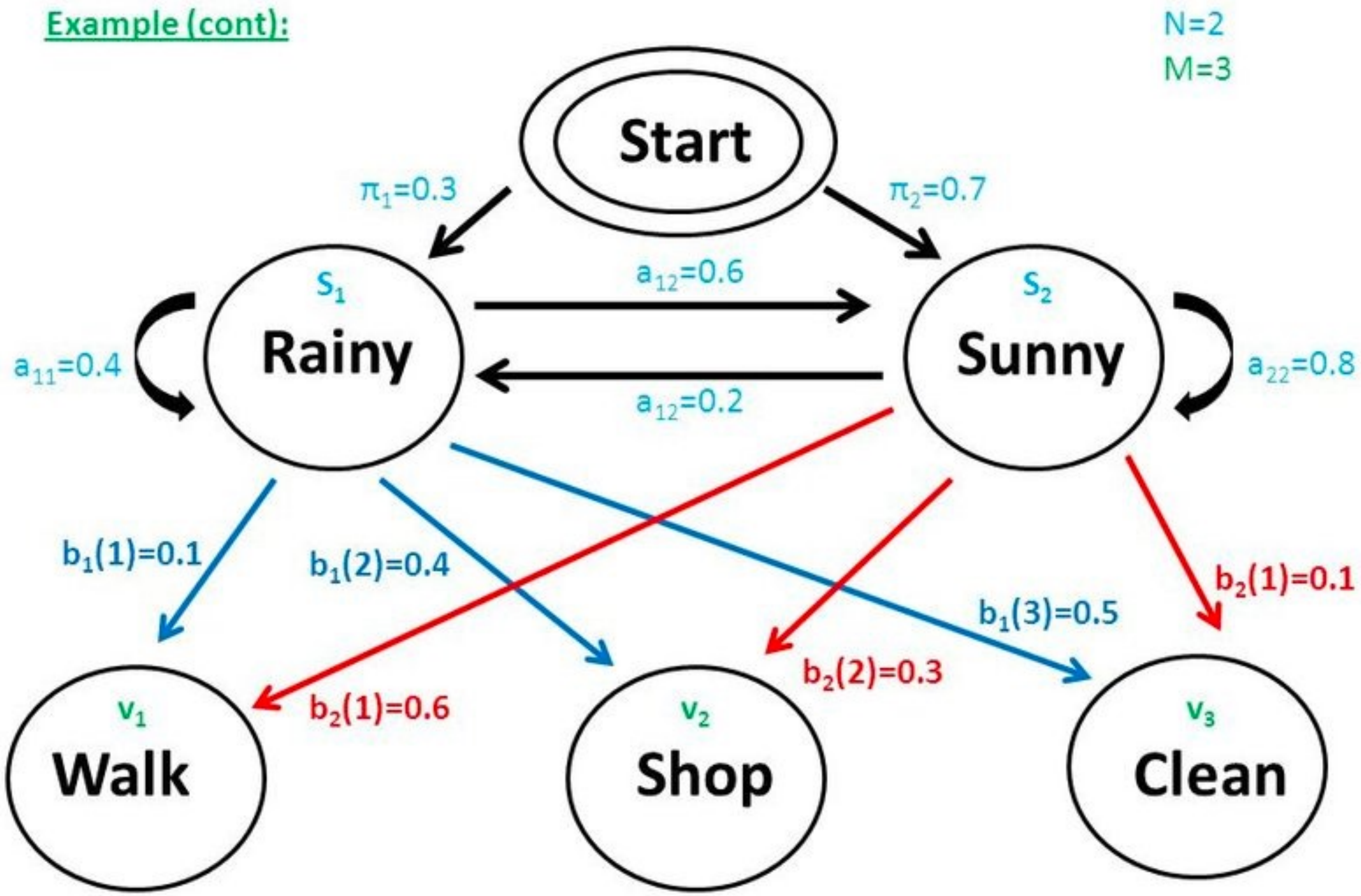




Graph neural network (GNN)



Hidden Markov Model



Basics

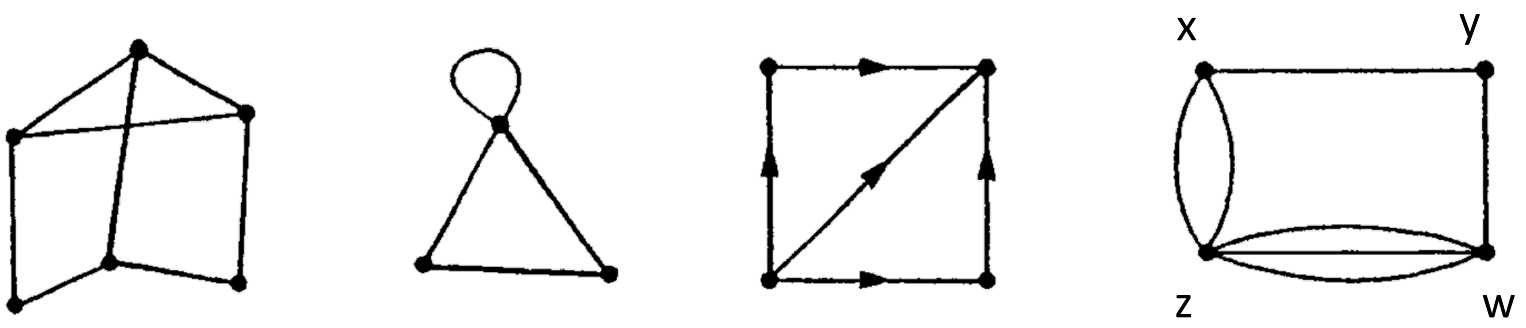
Graphs

- Definition A graph G is a pair (V, E)
  - V: set of vertices
  - E: set of edges
  - $e \in E$ corresponds to a pair of endpoints $x, y \in V$

We mainly focus on
Simple graph:
No loops, no multi-edges

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

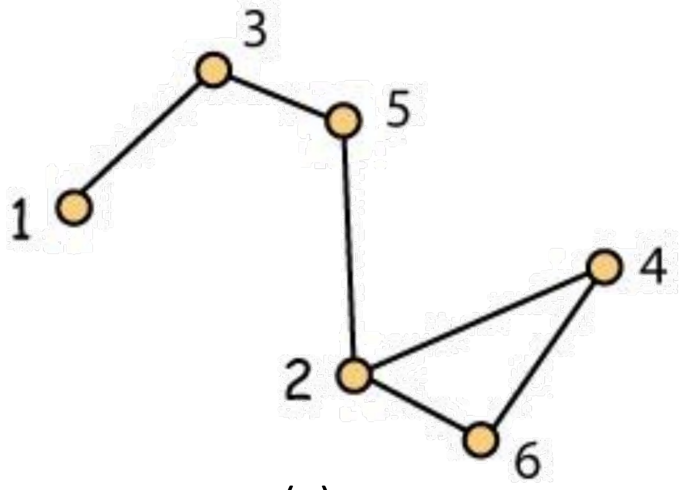


(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

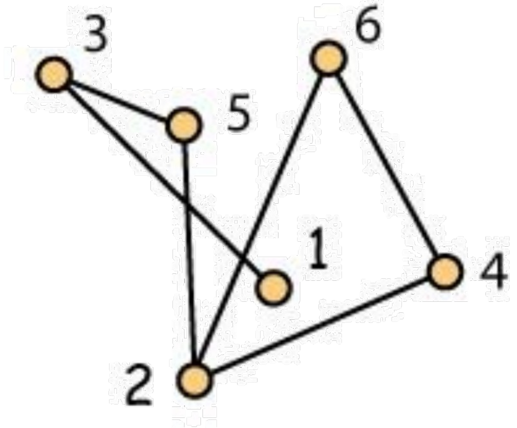
Figure 1.2

Graphs: All about adjacency

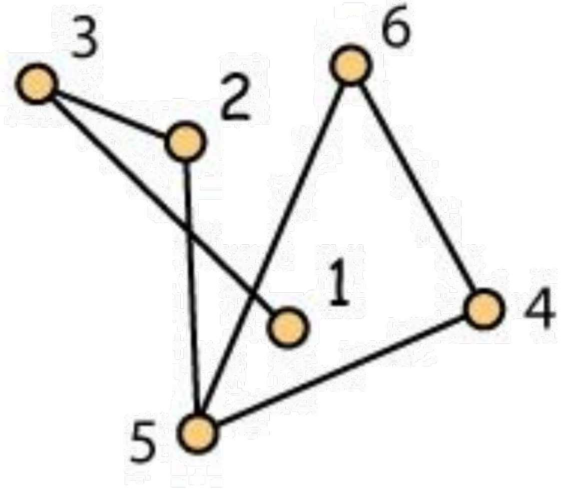
- Same graph or not



(a)



(b)



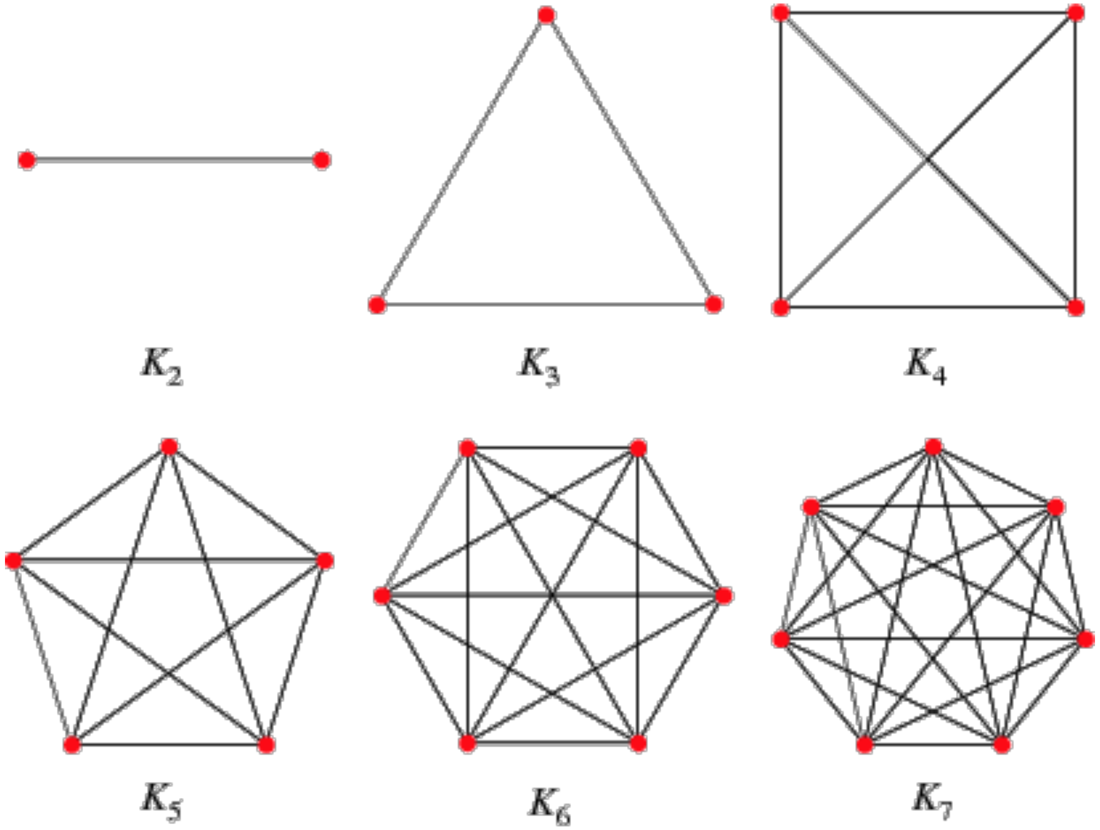
(c)

- Two graphs $G_1 = (V_1, E_1), G_1 = (V_2, E_2)$ are isomorphic if there is a bijection $f: V_1 \to V_2$ s.t.

$$e = \{a, b\} \in E_1 \Leftrightarrow f(e) := \{f(a), f(b)\} \in E_2$$

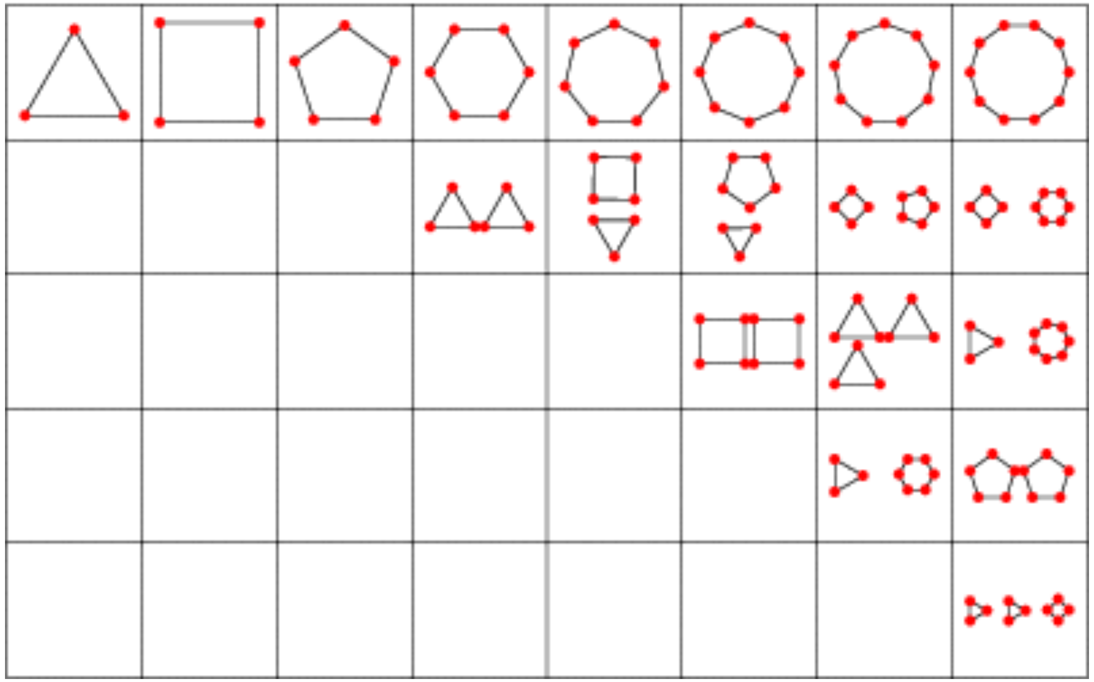
Example: Complete graphs

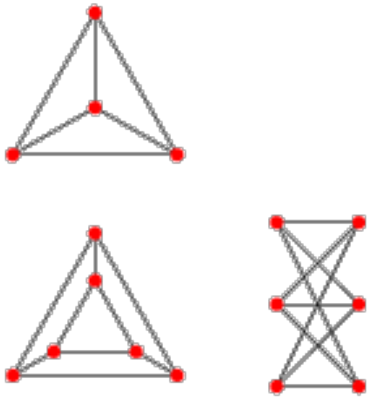
- There is an edge between every pair of vertices K_n

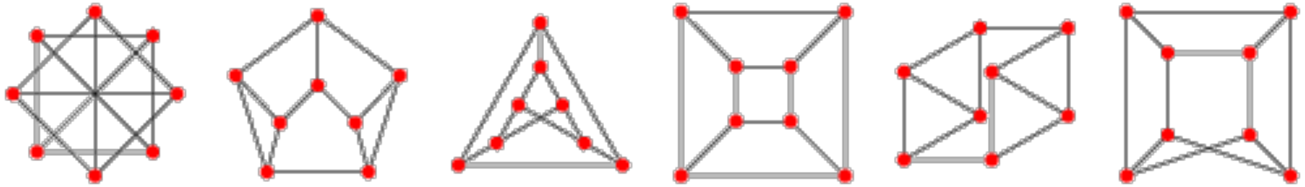


Example: Regular graphs

- Every vertex has the same degree

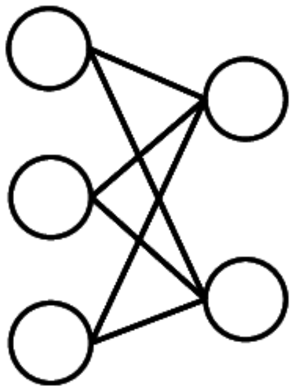




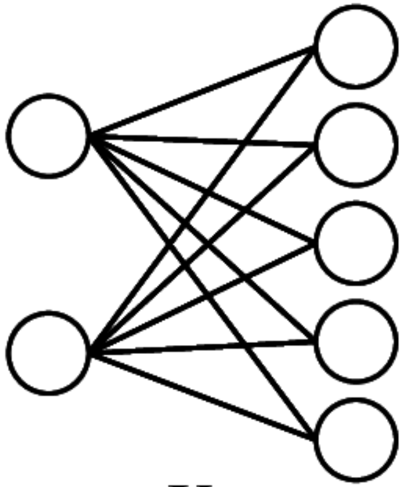


Example: Bipartite graphs

- The vertex set can be partitioned into two sets X and Y such that every edge in G has one end vertex in X and the other in Y
- Complete bipartite graphs



$\mathbf{K_{3,2}}$



$\mathbf{K_{2,5}}$

Example (1A, L): Peterson graph

- Show that the following two graphs are same/isomorphic

$n = 10$

$m = 15$

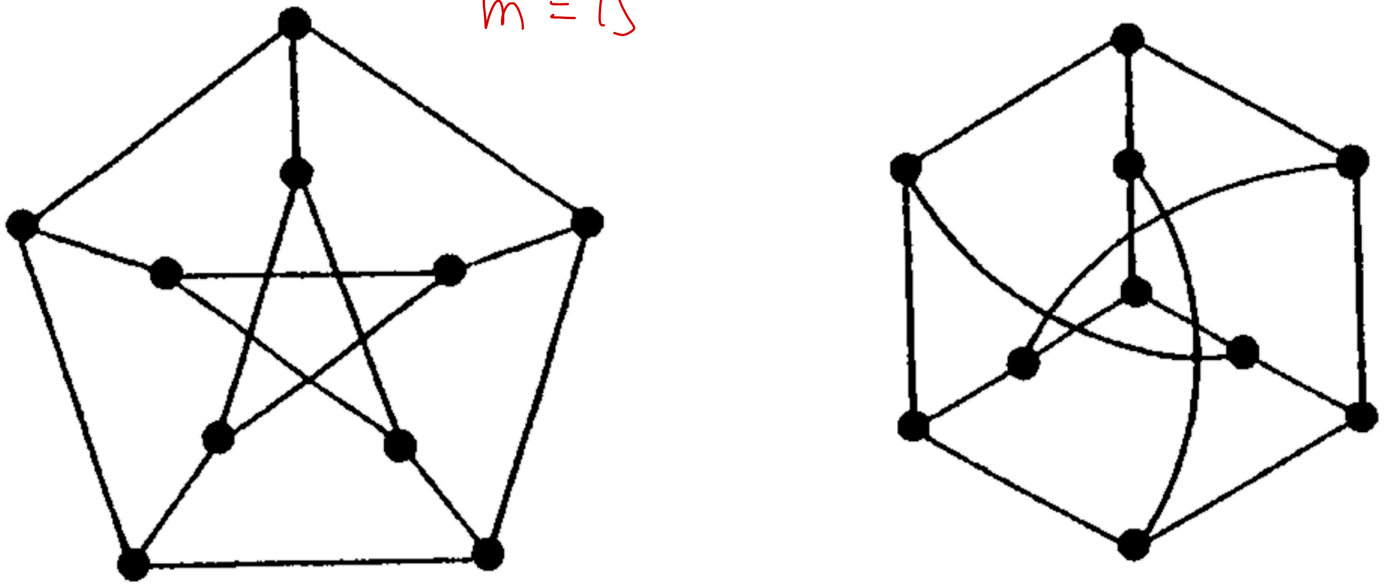
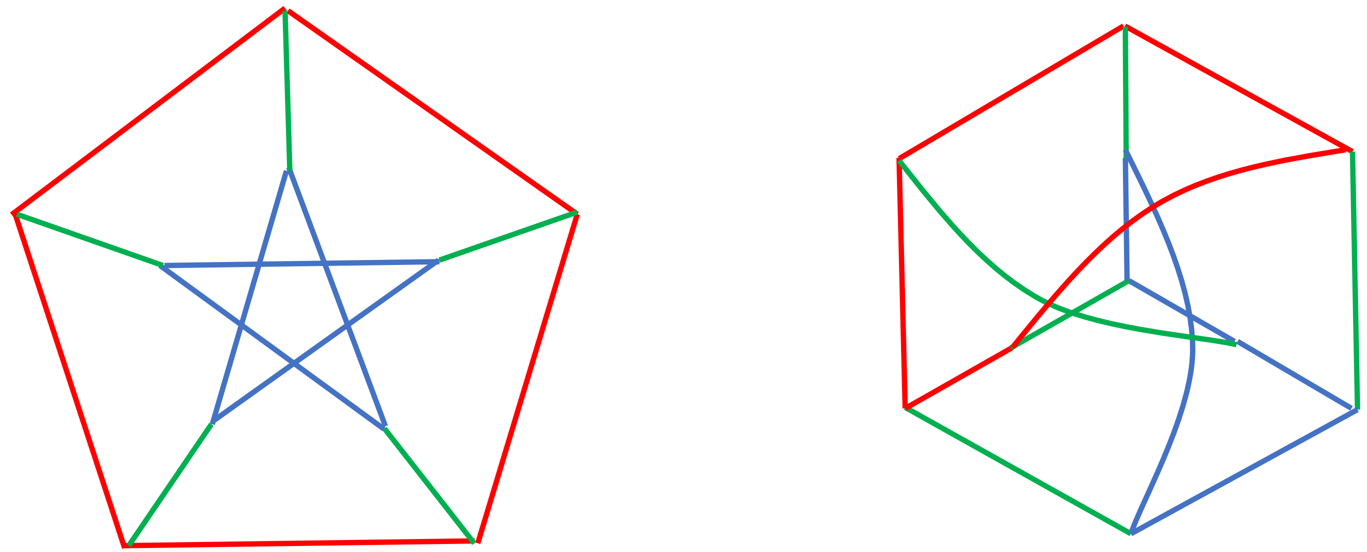


Figure 1.4

Example: Peterson graph (cont.)

- Show that the following two graphs are same/isomorphic

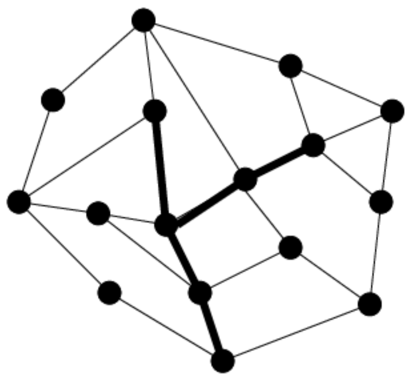


Subgraphs

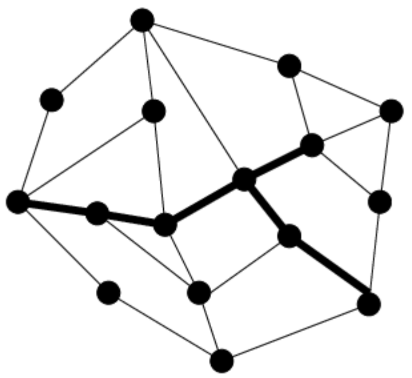
- A subgraph of a graph G is a graph H such that

$$V(H) \subseteq V(G), E(H) \subseteq E(G)$$

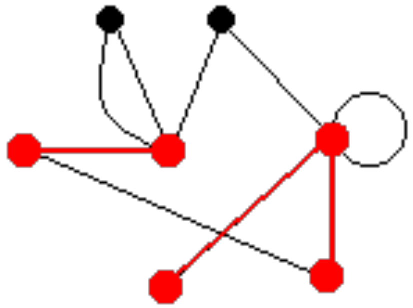
  and the ends of an edge $e \in E(H)$ are the same as its ends in G
  - H is a spanning subgraph when $V(H) = V(G)$
  - The subgraph of G induced by a subset $S \subseteq V(G)$ is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S



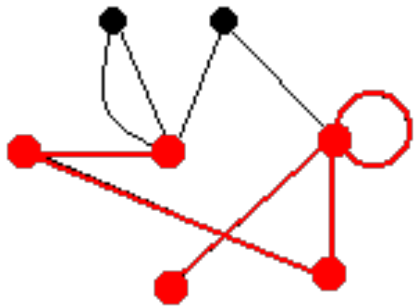
(a)



(b)



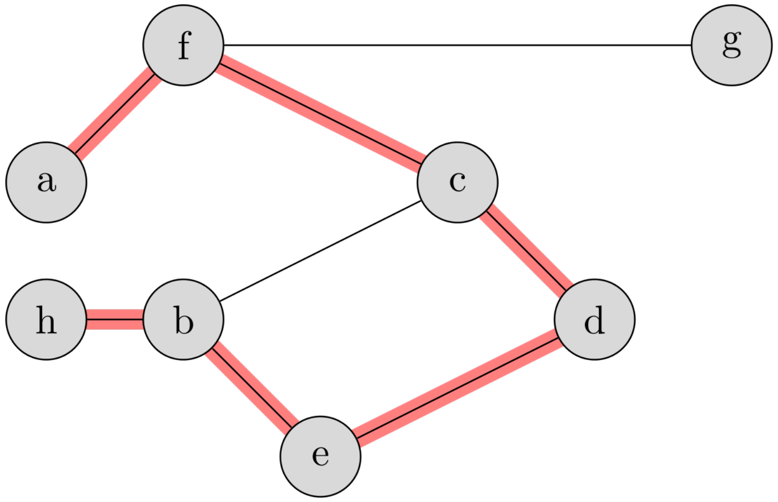
Subgraph (in red)



Induced Subgraph

Paths (路径)

- A path is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$ where vertices are all distinct
  - Or it can be written as $v_0 v_1 \dots v_k$ in simple graphs
- P^k: path of length k (the number of edges)



Walk (游走)

- A walk is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \ldots e_k v_k$
  - The vertices not necessarily distinct
  - The length = the number of edges
- Proposition (1.2.5, W) Every u-v walk contains a u-v path

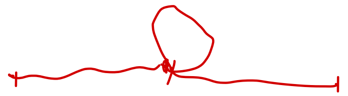
By induction

$k=0$ $u=v$ $\circ$

$\leq k$ ✓ u-v walk of $(k+1)$ length W

– distinct ✓

–



u-v walk $W' = v_0 \cdots v_{\ell_1} v_{\ell_2+1} \cdots v_{k+1}$ of length $\leq k$, with $v_0 = u$, $v_{k+1} = v$, $\subseteq W$

$\Rightarrow \exists$ u-v path $P \subseteq W' \subseteq W$

Cycles (环)

By induction,

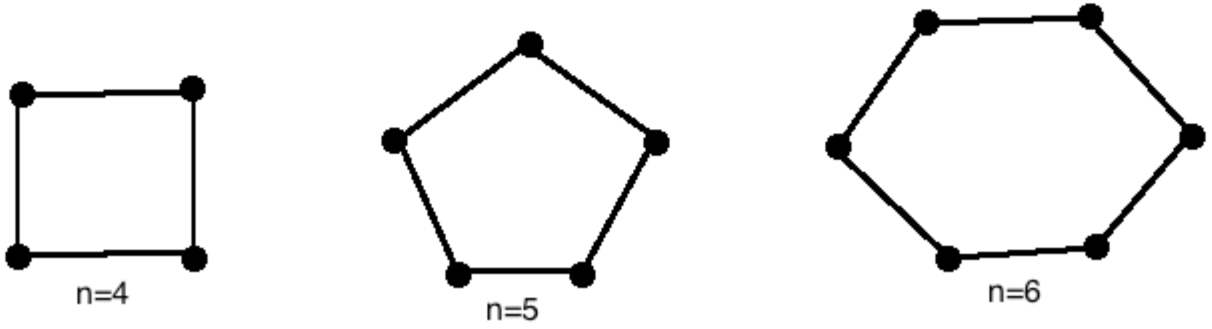
$k=3$ ✓

$\leq k$ closed walk of length $k+1$ (odd)

- distinct ✓

-

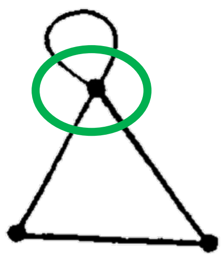
- If $P = x_0 x_1 \dots x_{k-1}$ is a path and $k \geq 3$, then the graph $C \coloneqq P + x_{k-1}x_0$ is called a cycle
- C^k: cycle of length k (the number of edges/vertices)



- Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Neighbors and degree

- Two vertices $a \neq b$ are called adjacent if they are joined by an edge
  - $N(x)$: set of all vertices adjacent to x
    - neighbors of x
  - A vertex is isolated vertex if it has no neighbors
- The number of edges incident with a vertex x is called the degree of x
  - A loop contributes 2 to the degree
- A graph is finite when both $E(G)$ and $V(G)$ are finite sets



graph with loop

Handshaking Theorem (Euler 1736)

- Theorem A finite graph G has an even number of vertices with odd degree

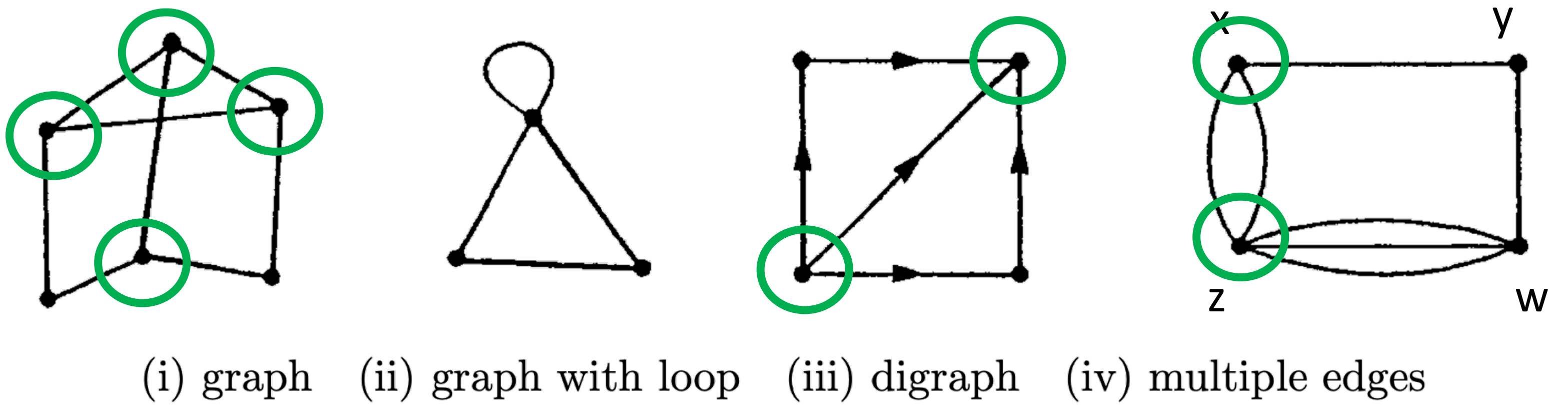


Figure 1.2

Proof

- Theorem A finite graph G has an even number of vertices with odd degree.
- Proof The degree of x is the number of times it appears in the right column. Thus

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

Degree

- Minimal degree of G: $\delta(G) = \min\{d(v) : v \in V\}$
- Maximal degree of G: $\Delta(G) = \max\{d(v) : v \in V\}$
- Average degree of G: $d(G) = \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{2|E|}{|V|}$

- All measure the `density' of a graph

- $d(G) \geq \delta(G) \geq$?

$d(G) = \frac{2|E|}{|V|}$

Degree (global to local)

$v \in G$ $\quad \frac{|E|}{|V|} < \frac{|E|-d(v)}{|V|-1} \Leftrightarrow d(v) < \frac{|E|}{|V|} = \frac{1}{2}d(G)$

$G = G_0 \supseteq G_1 \supseteq \dots \supseteq \dots$

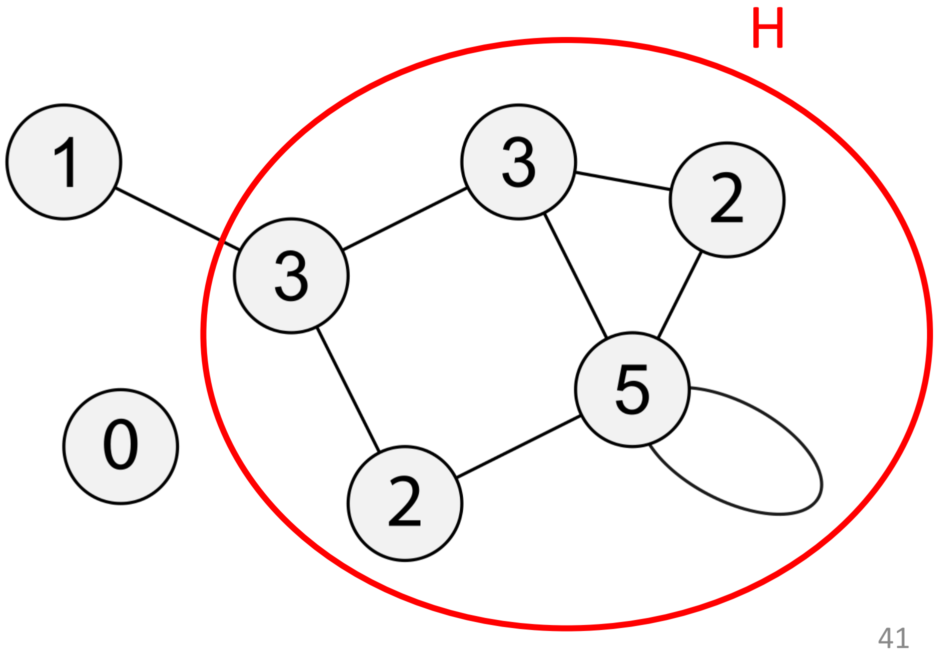
- G_i has vertex v_i s.t. $deg_{G_i}(v_i) \le \frac{1}{2}d(G_i) \Rightarrow G_{i+1} = G_i \setminus \{v_i\} \Rightarrow d(G_{i+1}) \ge d(G_i)$
- G_i has no such vertex $\quad \delta(G_i) > \frac{1}{2}d(G_i) \ge 0$

$(K_1.\ d(K_1) = 0.\ H \ne \phi)$

- Proposition (1.2.2, D) Every graph G with at least one edge has a subgraph H with

$$\delta(H) > \frac{1}{2}d(H) \ge \frac{1}{2}d(G)$$

- Example: $|G| = 7, d(G) = \frac{16}{7}$
- $\delta(H) = 2, d(H) = \frac{14}{5}$



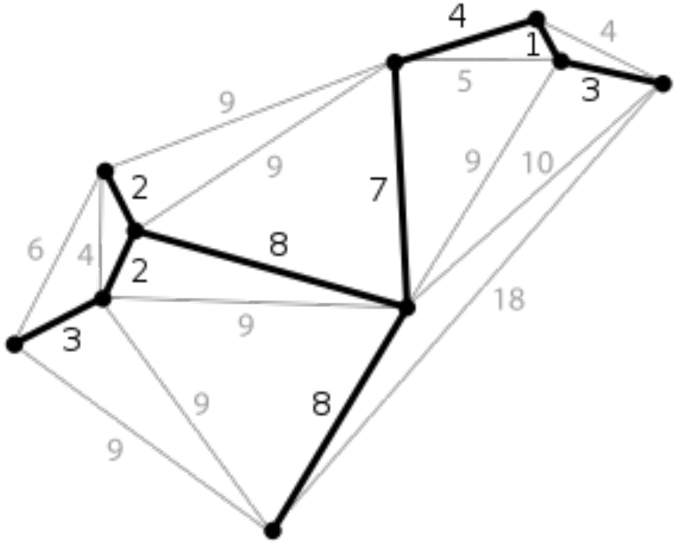
Minimal degree guarantees long paths and cycles

- Proposition (1.3.1, D) Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$, provided $\delta(G) \geq 2$.

Take the longest path P

$P = v_0 v_1 \cdots v_i \cdots v_k$ ← w

Consider all neighbors of v_k



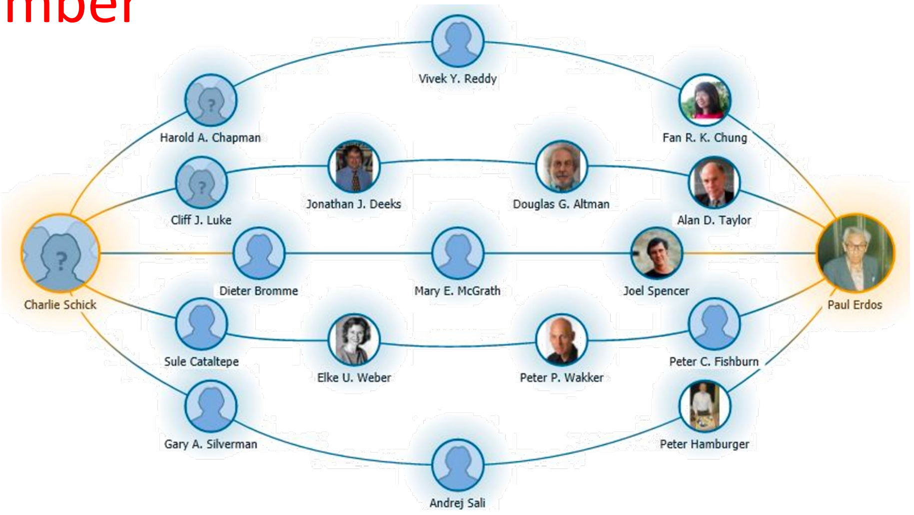
Distance and diameter

- The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x \sim y$ path
  - if no such path exists, we set $d(x, y) \coloneqq \infty$
- The greatest distance between any two vertices in G is the diameter of G

$$\operatorname{diam}(G) = \max_{x,y \in V} d(x, y)$$

Example -- Erdős number


- A well-known graph
  - vertices: mathematicians of the world
  - Two vertices are adjacent if and only if they have published a joint paper
  - The distance in this graph from some mathematician to the vertex Paul Erdős is known as his or her Erdős number



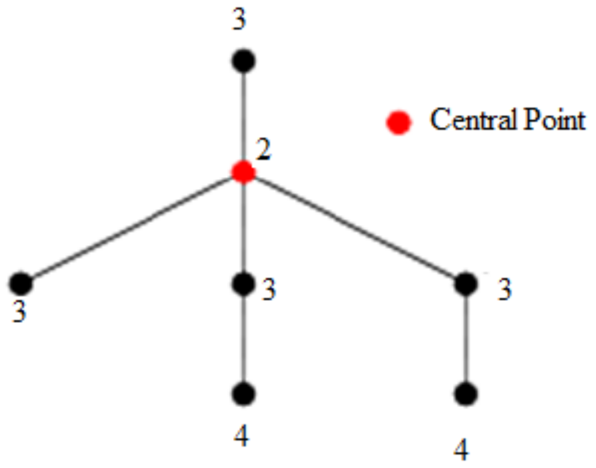
Radius and diameter

- A vertex is central in G if its greatest distance from other vertex is smallest, such greatest distance is the radius of G

$$\mathrm{rad(G)} := \min_{x \in V} \max_{y \in V} d(x, y)$$

$$\mathrm{diam}(G) = \max_{x} \max_{y} d(x,y)$$

- Proposition (1.4, H; Ex1.6, D) $\mathrm{rad}(G) \leq \mathrm{diam}(G) \leq 2\,\mathrm{rad}(G)$



Radius = 2
Diameter = 4

$\forall x, y \quad d(x,y) \leq d(x,z) + d(z,y)$

Take a center z $\quad \leq 2\,\mathrm{rad}(G)$

Radius and maximum degree control graph size

- **Proposition** (1.3.3, D) A graph G with radius at most r and maximum degree at most $\Delta \geq 3$ has fewer than $\frac{\Delta}{\Delta-2}(\Delta-1)^r$.

Take z as the center point

$D_i := \{x \in G : d(x,z) = i\}$

$|D_0| = 1$

$G = \bigcup_{i=0}^{r} D_i$

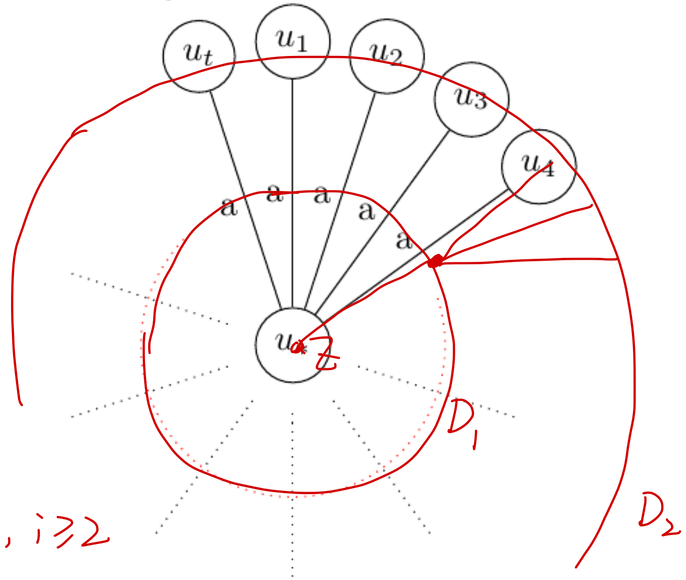
$|D_1| \leq \Delta$

$|D_2| \leq (\Delta-1) \cdot |D_1| \leq \Delta(\Delta-1)$

$|D_i| \leq (\Delta-1)|D_{i-1}| \leq (\Delta-1)^{i-1}\Delta, \; i \geq 2$

$|G| \leq 1 + \sum_{i=1}^{r} \Delta \cdot (\Delta-1)^{i-1} = \cdots$

Figure 1: Star Graph



Summary

Shuai Li

https://shuaili8.github.io

- Motivation and applications
- Basic concepts:
  - graph, isomorphism, subgraphs, paths, walks, cycles,
  - Neighbors, degree, distance, diameter, radius
- Examples:
  - Complete/regular/bipartite graphs, Peterson graph
- Theorems:
  - Handshaking
  - Large average degree guarantees dense subgraphs
  - Large minimal degree guarantees long paths and cycles
  - Radius and maximum degree control graph size

Questions?