Lecture 5: Matchings

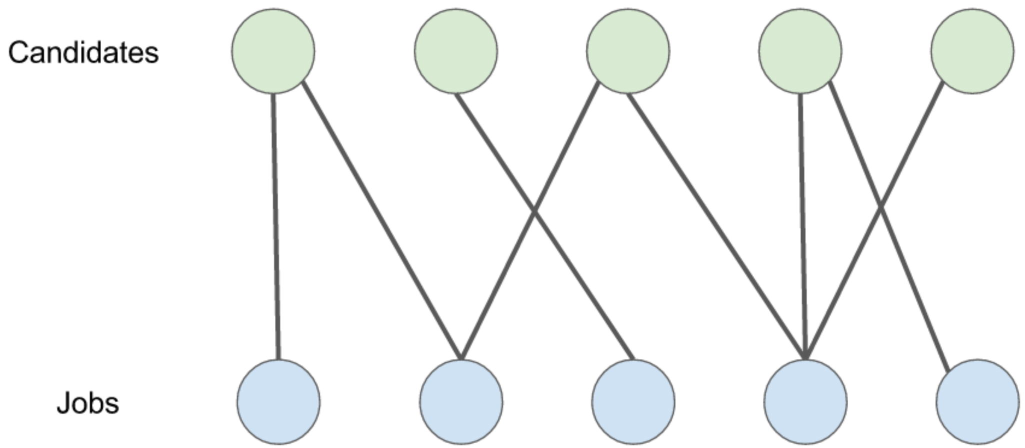
Shuai Li

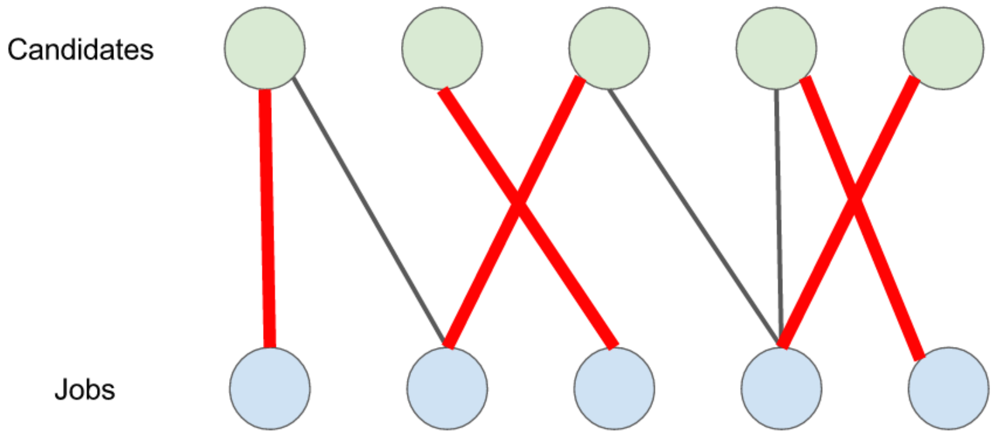
John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS3330/index.html

Motivating example





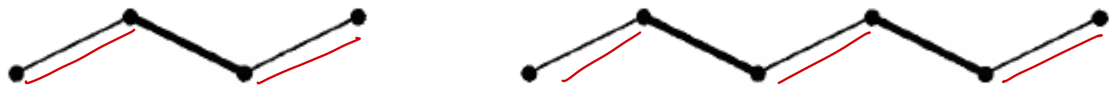
Definitions

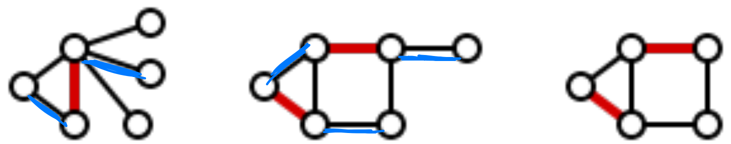
- A matching is a set of independent edges, in which no pair of edges shares a vertex
- The vertices incident to the edges of a matching M are M-saturated (饱和的); the others are M-unsaturated
- A perfect matching in a graph is a matching that saturates every vertex
- Example (3.1.2, W) The number of perfect matchings in $K_{n,n}$ is $n!$
- Example (3.1.3, W) The number of perfect matchings in K_{2n} is

$$f_n = (2n-1)(2n-3)\cdots 1 = (2n-1)!!$$

Maximal/maximum matchings 极大/最大

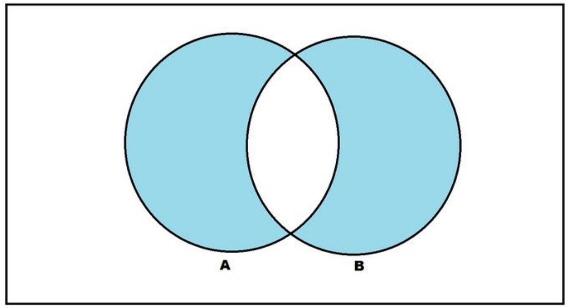
- A maximal matching in a graph is a matching that cannot be enlarged by adding an edge
- A maximum matching is a matching of maximum size among all matchings in the graph
- Example: P_3, P_5



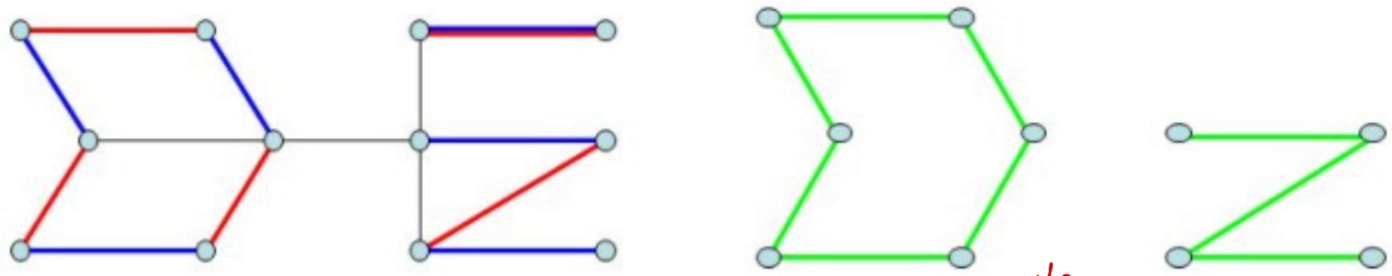


- Every maximum matching is maximal, but not every maximal matching is a maximum matching

Symmetric difference of matchings



- The symmetric difference of M, M' is $M \Delta M' = (M - M') \cup (M' - M)$
- Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



$\forall v \ d_M(v) \leq 1$

$F = M \Delta M' \quad d_F(v) \leq 2 \Rightarrow \Delta(F) \leq 2$

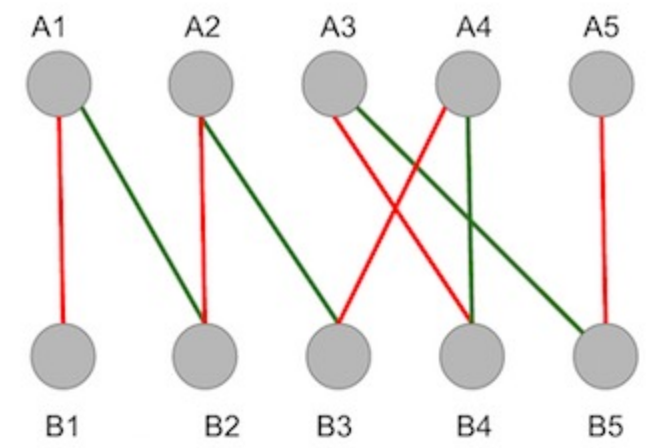
$\therefore$ Every component in F must be a path or a cycle

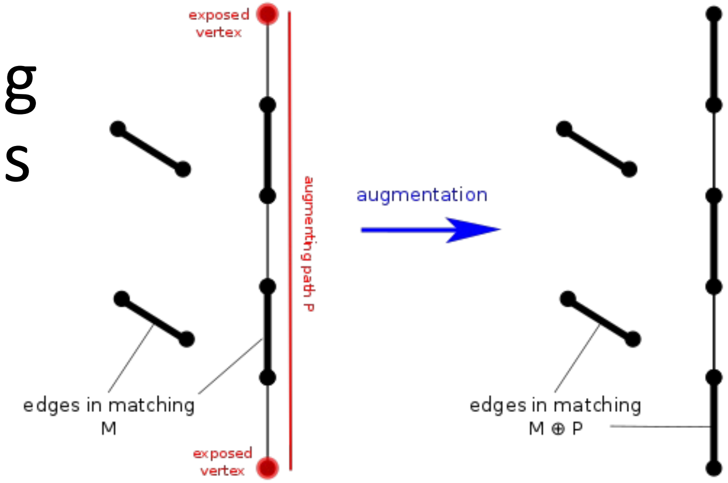
Every path or cycle alternates edges in M and M'

$\Rightarrow$ Every cycle must have even length

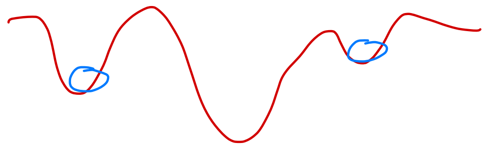
Maximum matching and augmenting path

- Given a matching M, an M-alternating path is a path that alternates between edges in M and edges not in M
- An M-alternating path whose endpoints are M-unsaturated is an M-augmenting path
- Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path



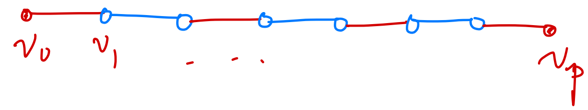


Lemma (3.1.9, W) Every component of the symmetric difference of two matchings is a path or an even cycle



maximum $\Leftrightarrow$ no M-augmenting path

"$\Rightarrow$" 反证法 $\exists$ M-augmenting path

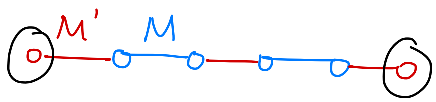


$M - \{v_1v_2, v_3v_4 \dots\} \cup \{v_0v_1, v_2v_3 \dots\}$ becomes a larger matching. 矛盾!

"$\Leftarrow$" If M is not maximum. take M' as the maximum.

$F = M \Delta M'$ even cycle, path

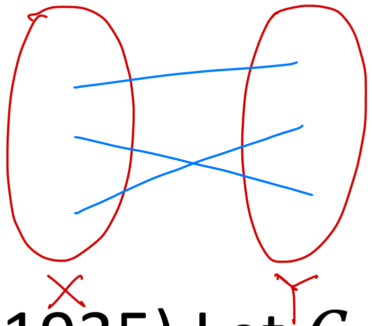
$\therefore$ There is a path component P w/ more edges from M'



Two endpoints must be M-unsaturated

P is an M-augmenting path

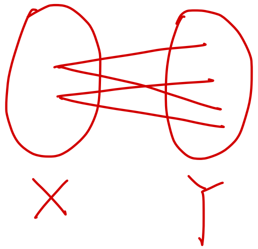
Hall's theorem (TONCAS)



- Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
  G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in $G \Leftrightarrow G$ has no M-augmenting path

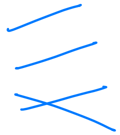
- Exercise. Read the other two proofs in Diestel.
- Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching



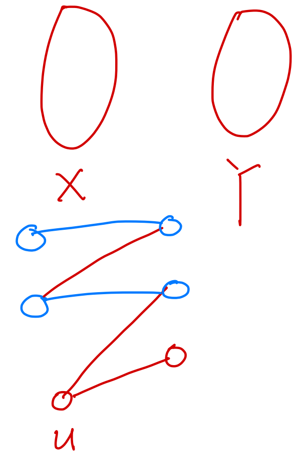
$k|X| = k|Y| \Rightarrow |X| = |Y|$

$\forall S \subseteq X,\ k|S| \leq k|N(S)| \Rightarrow |S| \leq |N(S)|$

"⇒" ✓



"⇐" 反证法. M is maximum but not matching of X



Let $u \in X$ be an M-unsaturated vertex.

$A = \{ v \in G : v$ can be reached by M-alternating paths from $u \}$

$S := A \cap X$

$T := A \cap Y$

- $S - \{u\}$, T contain only M-saturated vertices
- $S - \{u\}$ and T are 1-1 correspondence by edges in M

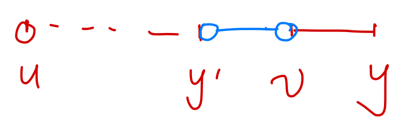
  $|T| = |S| - 1$

- $N(S) = T$

  $T \subseteq N(S)$. $\exists y \in N(S) \setminus T$. $\Rightarrow \exists v \in S$, $vy \in E$

  $v = u \Rightarrow y \in T$

  $v \neq u \Rightarrow$



M-alternating path

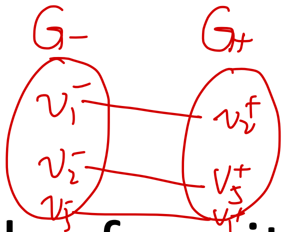
$\Rightarrow y \in T$

General regular graph

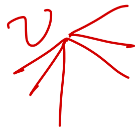
G is 2k-regular. G is connected (w.l.o.g)

$\Rightarrow$ G is Eulerian $\Rightarrow$ $v_1 e_1 v_2 e_2 \cdots v_n e_n v_{n+1} = v_1$

$v_1^+ v_1^- \quad v_2^+ v_2^-$



$G = \text{Bipartite}(G_-, G_+)$ is k-regular



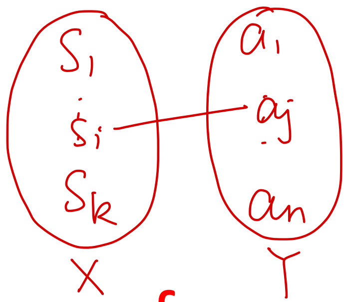
- Corollary (2.1.5, D) Every regular graph of positive even degree has a 2-factor
  - A k-regular spanning subgraph is called a k-factor
  - A perfect matching is a 1-factor

Theorem (1.2.26, W) A graph G is Eulerian $\Leftrightarrow$ it has at most one nontrivial component and its vertices all have even degree

Corollary (3.1.13, W; 2.1.3, D) Every k-regular ($k > 0$) bipartite graph has a perfect matching

Application to SDR

$\mathcal{S} = \{S_1, \ldots, S_k\}$ $\quad \cup \mathcal{S} = \{a_1, \ldots, a_n\}$



$S_i a_j \in E \Leftrightarrow a_j \in S_i$

SDR $\Leftrightarrow \exists$ a matching of X into Y

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$S_1 = \{2, 8\}$,
$S_2 = \{8\}$,
$S_3 = \{5, 7\}$,
$S_4 = \{2, 4, 8\}$,
$S_5 = \{2, 4\}$.

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \ldots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

> Theorem (3.1.11, W; 1.51, H; 2.1.2, D; Hall 1935) Let G be a bipartite graph with partition X, Y.
> G contains a matching of $X \Leftrightarrow |N(S)| \geq |S|$ for all $S \subseteq X$

König Theorem
Augmenting Path Algorithm

Vertex cover

- A set $U \subseteq V$ is a (vertex) cover of E if every edge in G is incident with a vertex in U
- Example:
  - Art museum is a graph with hallways are edges and corners are nodes
  - A security camera at the corner will guard the paintings on the hallways
  - The minimum set to place the cameras?

König-Egeváry Theorem (Min-max theorem)

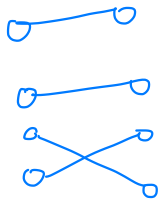
- Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

> Theorem (3.1.10, W; 1.50, H; Berge 1957) A matching M in a graph G is a maximum matching in G $\Leftrightarrow$ G has no M-augmenting path

$\forall$ matching M. vertex cover C.

optimal

matching | vertex cover

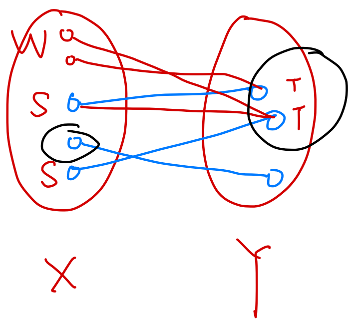


$|M| \le |C|$

"="

$\le$

$G = B(X, Y)$ Let M be a maximum matching



$W :=$ M-unsaturated points in X

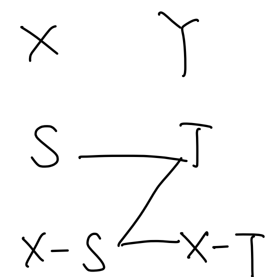
$A := \{$ vertices in G that can be reached via M-alternating paths from $W \}$

$S := A \cap X$

$T := A \cap Y$

- $S - W$ and T are 1-1 correspondence via matchings in M

$|S| - |W| = |T|$



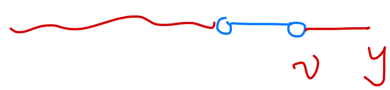
$|W| = |X| - |M|$

- $N(S) = T$

$T \subseteq N(S)$. $\forall y \in N(S) - T$, $\exists v \in S$, $vy \in E$

$v \in W$

$v \notin W$



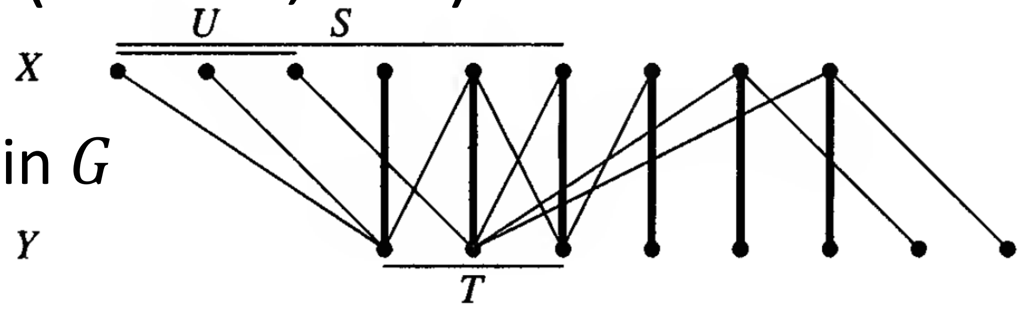
M-alternating path

- $C = (X - S) \cup T$ is vertex cover

$\Leftrightarrow$ No edges between S and $X - T$

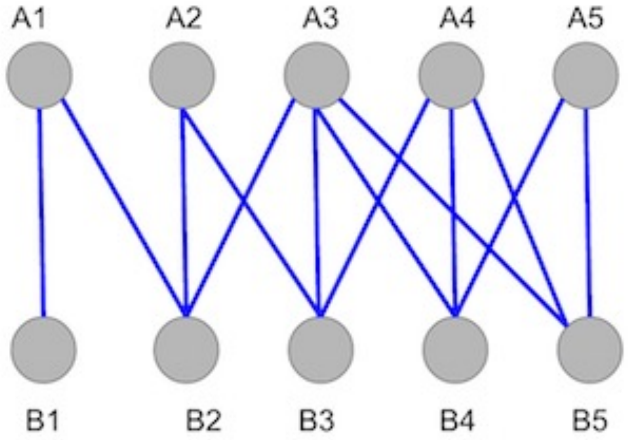
$|C| = |X| - |S| + |T| = |X| - |W| = |M|$

Augmenting path algorithm (3.2.1, W)

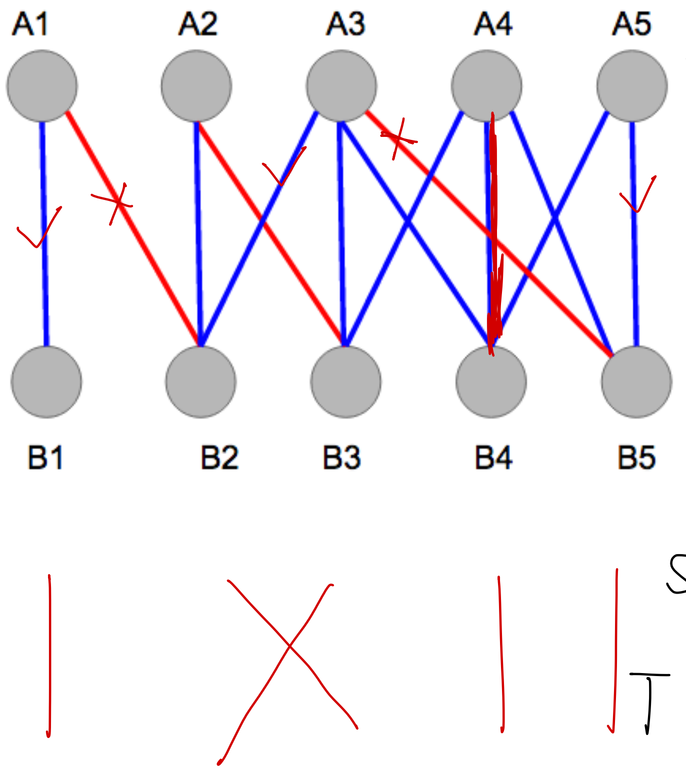


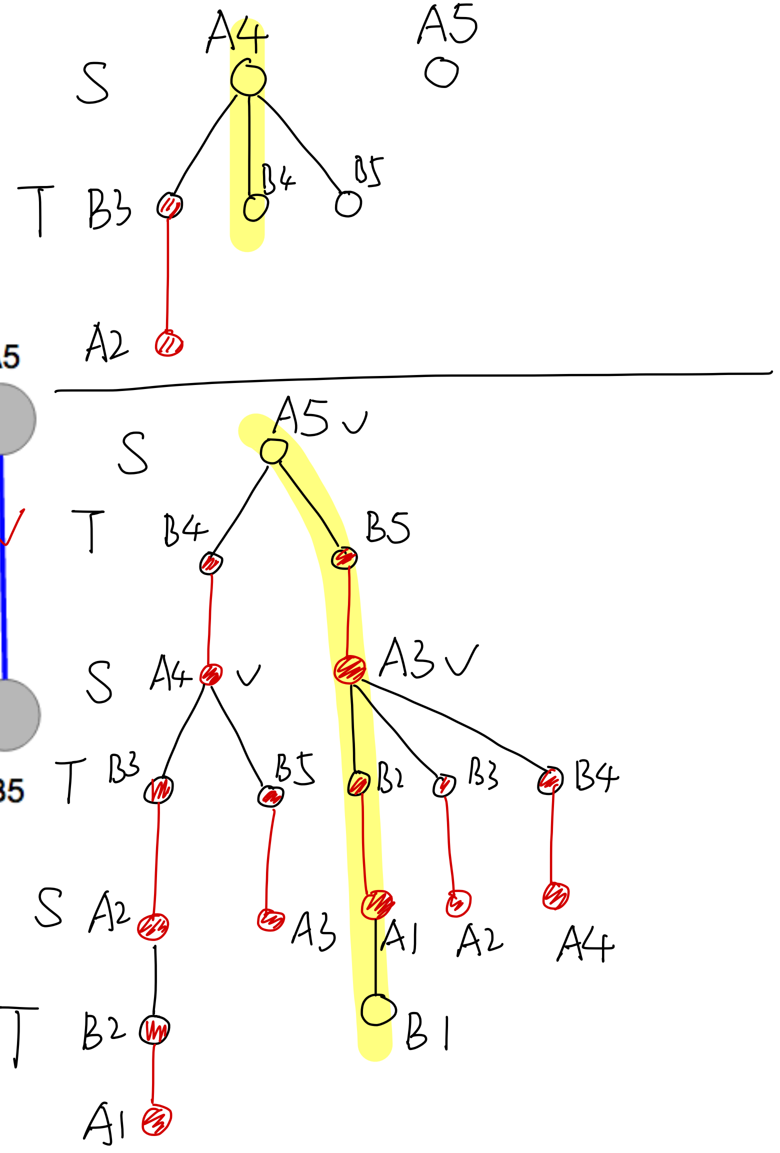
- **Input**: G is Bipartite with X, Y, a matching M in G
  $U = \{M\text{–unsaturated vertices in } X\}$
- **Idea**: Explore M-alternating paths from U letting $S \subseteq X$ and $T \subseteq Y$ be the sets of vertices reached
- **Initialization**: $S = U, T = \emptyset$ and all vertices in S are unmarked
- **Iteration**:
  - If S has no unmarked vertex, stop and report $T \cup (X - S)$ as a minimum cover and M as a maximum matching
  - Otherwise, select an unmarked $x \in S$ to explore
    - Consider each $y \in N(x)$ such that $xy \notin M$
      - If y is unsaturated, terminate and report an M-augmenting path from U to y
      - Otherwise, $yw \in M$ for some w
        - include y in T (reached from x) and include w in S (reached from y)
    - After exploring all such edges incident to x, mark x and iterate.

Example



Red: A random matching





Theoretical guarantee for Augmenting path algorithm

- Theorem (3.2.2, W) Repeatedly applying the Augmenting Path Algorithm to a bipartite graph produces a matching and a vertex cover of equal size

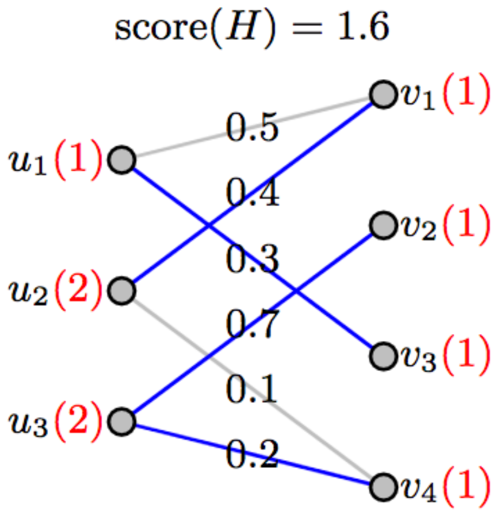
Weighted Bipartite Matching
Hungarian Algorithm

Weighted bipartite matching

- The maximum weighted matching problem is to seek a perfect matching M to maximize the total weight $w(M)$
- Bipartite graph
  - W.l.o.g. Assume the graph is $K_{n,n}$ with $w_{i,j} \geq 0$ for all $i, j \in [n]$
  - Optimization:

$$\begin{aligned} \max \quad & w(M_a) = \sum_{i,j} a_{i,j} w_{i,j} \\ s.t. \quad & a_{i,1} + \cdots + a_{i,n} = 1 \text{ for any } i \\ & a_{1,j} + \cdots + a_{n,j} = 1 \text{ for any } j \\ & a_{i,j} \in \{0,1\} \end{aligned}$$

  - Integer programming
  - General IP problems are NP-Complete



(Weighted) cover

- A (weighted) cover is a choice of labels $u_1, \dots, u_n$ and $v_1, \dots, v_n$ such that $u_i + v_j \geq w_{i,j}$ for all i, j
  - The cost $c(u, v)$ of a cover (u, v) is $\sum_i u_i + \sum_j v_j$
  - The minimum weighted cover problem is that of finding a cover of minimum cost
- Optimization problem

$$
\begin{aligned}
\min \quad & c(u, v) = \sum_i u_i + \sum_j v_j \\
s.t. \quad & u_i + v_j \geq w_{i,j} \text{ for any } i, j
\end{aligned}
$$

Duality

(IP)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\ a_{i,1} + \cdots + a_{i,n} = 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} = 1 \text{ for any } j$$

$$a_{i,j} \in \{0,1\}$$

(Linear programming)

$$\max \sum_{i,j} a_{i,j} w_{i,j}$$

$$s.t.\ a_{i,1} + \cdots + a_{i,n} = 1 \text{ for any } i$$

$$a_{1,j} + \cdots + a_{n,j} = 1 \text{ for any } j$$

$$a_{i,j} \geq 0$$

(Dual)

$$\min \sum_{i} u_i + \sum_{j} v_j$$

$$s.t.\ u_i + v_j \geq w_{i,j} \text{ for any } i,j$$

- Weak duality theorem
  - For each feasible solution a and (u, v)

$$\sum_{i,j} a_{i,j} w_{i,j} \leq \sum_{i} u_i + \sum_{j} v_j$$

thus $\max \sum_{i,j} a_{i,j} w_{i,j} \leq \min \sum_i u_i + \sum_j v_j$

Duality (cont.)

- Strong duality theorem
  - If one of the two problems has an optimal solution, so does the other one and that the bounds given by the weak duality theorem are tight

$$\max \sum_{i,j} a_{i,j} w_{i,j} = \min \sum_i u_i + \sum_j v_j$$

- Lemma (3.2.7, W) For a perfect matching M and cover (u, v) in a weighted bipartite graph G, $c(u, v) \geq w(M)$.
  $c(u, v) = w(M) \Leftrightarrow M$ consists of edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  In this case, M and (u, v) are optimal.

Equality subgraph

- The equality subgraph $G_{u,v}$ for a cover (u, v) is the spanning subgraph of $K_{n,n}$ having the edges $x_i y_j$ such that $u_i + v_j = w_{i,j}$
  - So if $c(u, v) = w(M)$ for some perfect matching M, then M is composed of edges in $G_{u,v}$
  - And if $G_{u,v}$ contains a perfect matching M, then (u, v) and M (whose weights are $u_i + v_j$) are both optimal

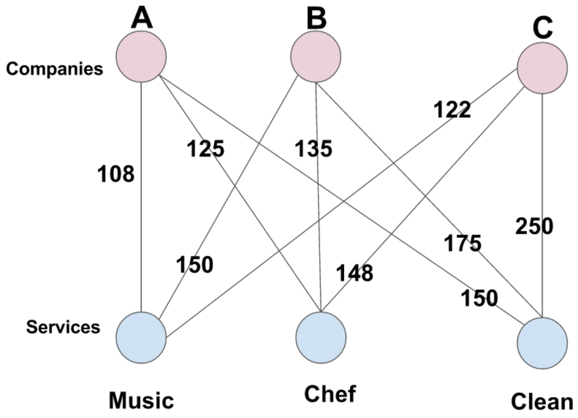
Hungarian algorithm

- **Input**: Weighted $K_{n,n} = B(X, Y)$
- **Idea**: Iteratively adjusting the cover (u, v) until the equality subgraph $G_{u,v}$ has a perfect matching
- **Initialization**: Let (u, v) be a cover, such as $u_i = \max_j w_{i,j}$, $v_j = 0$

(Dual)

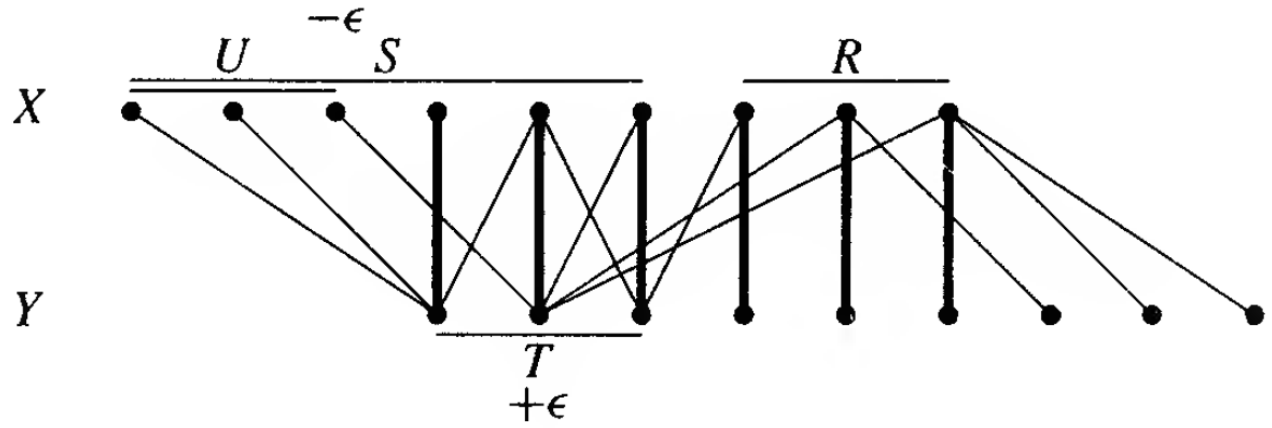
$$\min \sum_i u_i + \sum_j v_j$$

$$s.t.\ \ u_i + v_j \geq w_{i,j} \text{ for any } i, j$$

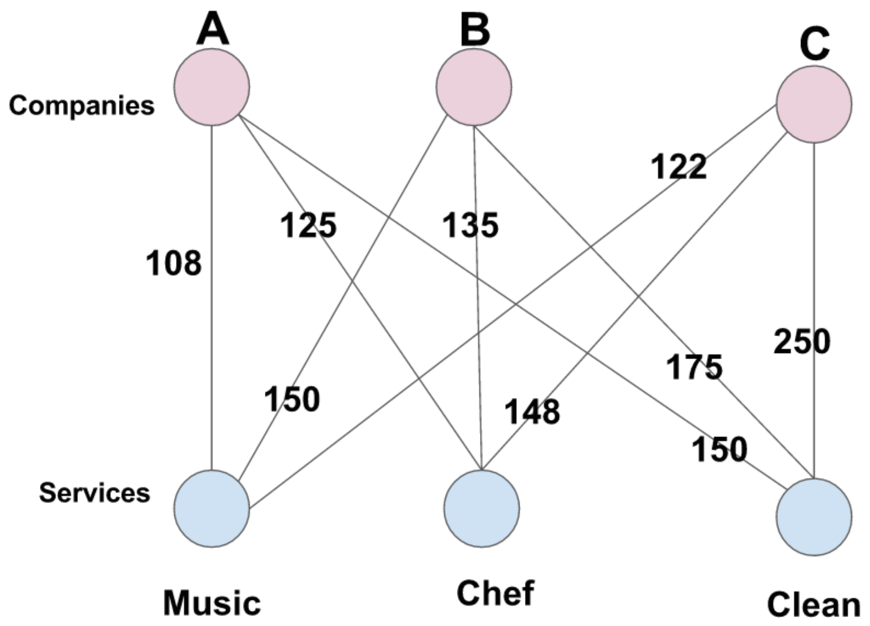


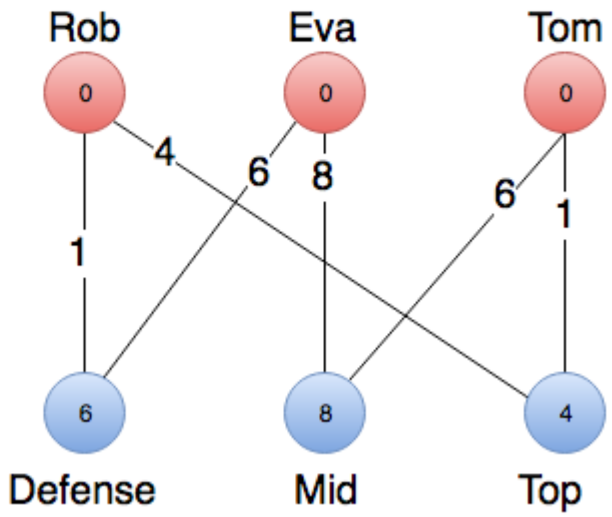
Hungarian algorithm (cont.)

- **Iteration**: Find a maximum matching M in $G_{u,v}$
  - If M is a perfect matching, stop and report M as a maximum weight matching
  - Otherwise, let Q be a vertex cover of size $|M|$ in $G_{u,v}$
    - Let $R = X \cap Q, T = Y \cap Q$
      $$\epsilon = \min\{u_i + v_j - w_{i,j} : x_i \in X - R, y_j \in Y - T\}$$
    - Decrease u_i by ϵ for $x_i \in X - R$ and increase v_j by ϵ for $y_j \in T$
  - Form the new equality subgraph and repeat



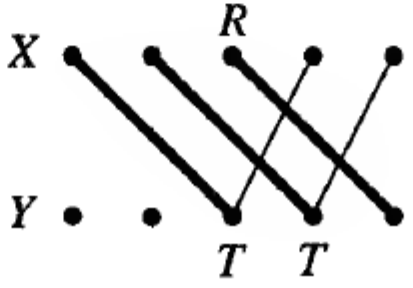
Example


A
B
C
Companies
108
125
135
122
250
175
150
148
150
Services
Music
Chef
Clean


Rob
Eva
Tom
0
0
0
4
6
8
6
1
1
6
8
4
Defense
Mid
Top

Example 2: Excess matrix

$$
\begin{pmatrix} 4 & 1 & 6 & 2 & 3 \\ 5 & 0 & 3 & 7 & 6 \\ 2 & 3 & 4 & 5 & 8 \\ 3 & 4 & 6 & 3 & 4 \\ 4 & 6 & 5 & 8 & 6 \end{pmatrix}
\rightarrow
\begin{array}{c} \\ 6 \\ 7 \\ 8 \\ 6 \\ 8 \\ \\ \end{array}
\begin{array}{c} 0 \quad 0 \quad 0 \quad 0 \quad 0 \\ \begin{pmatrix} 2 & 5 & \underline{0} & 4 & 3 \\ 2 & 7 & 4 & \underline{0} & 1 \\ 6 & 5 & 4 & 3 & \underline{0} \\ 3 & 2 & 0 & 3 & 2 \\ 4 & 2 & 3 & 0 & 2 \end{pmatrix} \\ \quad\quad\quad T \quad T \quad \end{array}
\begin{array}{c} \\ \\ R \\ \\ \\ \end{array}
$$



$$
\begin{array}{c} \\ 5 \\ 6 \\ 8 \\ 5 \\ 7 \\ \\ \end{array}
\begin{array}{c} 0 \quad 0 \quad 1 \quad 1 \quad 0 \\ \begin{pmatrix} 1 & 4 & \underline{0} & 4 & 2 \\ 1 & 6 & 4 & \underline{0} & 0 \\ 6 & 5 & 5 & 4 & \underline{0} \\ 2 & 1 & 0 & 3 & 1 \\ 3 & 1 & 3 & 0 & 1 \end{pmatrix} \\ \quad\quad\quad T \quad T \quad T \end{array}
$$

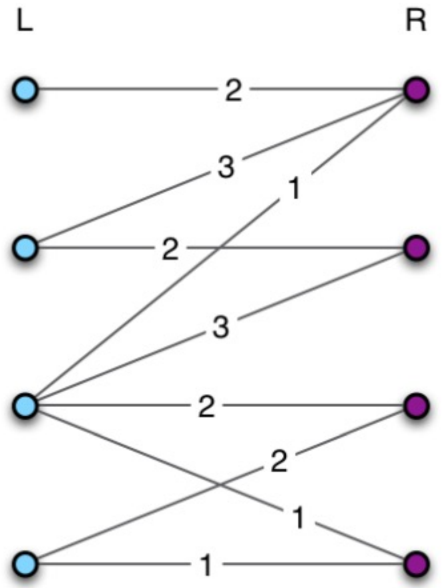
$$
\rightarrow
\begin{array}{c} \\ 4 \\ 5 \\ 7 \\ 4 \\ 6 \\ \end{array}
\begin{array}{c} 0 \quad 0 \quad 2 \quad 2 \quad 1 \\ \begin{pmatrix} 0 & 3 & \underline{0} & 4 & 2 \\ \underline{0} & 5 & 4 & 0 & 0 \\ 5 & 4 & 5 & 4 & \underline{0} \\ 1 & \underline{0} & 0 & 3 & 1 \\ 2 & 0 & 3 & \underline{0} & 1 \end{pmatrix} \end{array}
$$

Optimal value is the same
But the solution is not unique

Theoretical guarantee for Hungarian algorithm

- Theorem (3.2.11, W) The Hungarian Algorithm finds a maximum weight matching and a minimum cost cover

Example 3



Back to (unweighted) bipartite graph

- The weights are binary $0,1$
- Hungarian algorithm always maintain integer labels in the weighted cover, thus the solution will always be $0,1$
- The vertices receiving label 1 must cover the weight on the edges, thus cover all edges
- So the solution is a minimum vertex cover

Summary

Shuai Li

https://shuaili8.github.io

Questions?

- Matching in bipartite graphs
  - Hall's Theorem (TONCAS)
  - König Theorem: For bipartite graph, the maximum size of a matching is equal to the minimum size of a vertex cover of its edges
  - Augmenting Path Algorithm
- Matchings in weighted bipartite graphs
  - Weighted cover, Hungarian algorithm, equality subgraph, excess matrix