Girth

• The minimum length of a cycle in a graph $G$ is the girth $g(G)$ of $G$

• Example: The Peterson graph is the unique 5-cage
  • cubic graph (every vertex has degree 3)
  • girth = 5
  • smallest graph satisfies the above properties
Girth (cont.)

• A tree has girth $\infty$
• Note that a tree can be colored with two different colors
• $\implies$ A graph with large girth has small chromatic number?
• Unfortunately NO!
• Theorem (Erdős, 1959) For all $k, l$, there exists a graph $G$ with $g(G) > l$ and $\chi(G) > k$. 
Girth and diameter

• Proposition (1.3.2, D) Every graph $G$ containing a cycle satisfies $g(G) \leq 2 \text{diam}(G) + 1$
Girth and minimal degree lower bounds

\[ n_0(\delta, g) := \begin{cases} 1 + \delta \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r + 1 \text{ is odd} \\ 2 \sum_{i=0}^{r-1} (\delta - 1)^i, & \text{if } g = 2r \text{ is even} \end{cases} \]

- **Exercise** (Ex7, ch1, D) Let \( G \) be a graph. If \( \delta(G) \geq \delta \geq 2 \) and \( g(G) \geq g \), then \( |G| \geq n_0(\delta, g) \)

- **Corollary** (1.3.5, D) If \( \delta(G) \geq 3 \), then \( g(G) < 2 \log |G| \)
Theorem (1.3.23, W, Mantel 1907) The maximum number of edges in an $n$-vertex triangle-free simple graph is $\lceil n^2/4 \rceil$

- The bound is best possible
- There is a triangle-free graph with $\lceil n^2/4 \rceil$ edges: $K_{\lfloor n/2 \rfloor, \lfloor n/2 \rfloor}$
- Extremal problems
Connected, connected component

- A graph $G$ is connected if $G \neq \emptyset$ and any two of its vertices are linked by a path
- A maximal connected subgraph of $G$ is a (connected) component
Quiz

• **Problem** (1B, L) Suppose $G$ is a graph on 10 vertices that is not connected. Prove that $G$ has at most 36 edges. Can equality occur?

• **More general** (Ex9, S1.1.2, H) Let $G$ be a graph of order $n$ that is not connected. What is the maximum size of $G$?
Connected vs. minimal degree

• Proposition (1.3.15, W) If $\delta(G) \geq \frac{n-1}{2}$, then $G$ is connected

• (Ex16, S1.1.2, H) (1.3.16, W)
  If $\delta(G) \geq \frac{n-2}{2}$, then $G$ need not be connected

• Extremal problems

• “best possible” “sharp”
Add/delete an edge

- Components are pairwise disjoint; no two share a vertex
- Adding an edge decreases the number of components by 0 or 1
  - $\Rightarrow$ deleting an edge increases the number of components by 0 or 1
- Proposition (1.2.11, W)
  Every graph with $n$ vertices and $k$ edges has at least $n - k$ components
- An edge $e$ is called a bridge if the graph $G - e$ has more components
- Proposition (1.2.14, W)
  An edge $e$ is a bridge $\iff$ $e$ lies on no cycle of $G$
  - Or equivalently, an edge $e$ is not a bridge $\iff$ $e$ lies on a cycle of $G$
Cut vertex and connectivity

• A node $v$ is a cut vertex if the graph $G - v$ has more components

• A proper subset $S$ of vertices is a vertex cut set if the graph $G - S$ is disconnected

• The connectivity, $\kappa(G)$, is the minimum size of a cut set of $G$
  • The graph is $k$-connected for any $k \leq \kappa(G)$
Connectivity properties

• $\kappa(K^n) = n - 1$

• If $G$ is disconnected, $\kappa(G) = 0$
  
  • $\Rightarrow$ A graph is connected $\iff \kappa(G) \geq 1$

• If $G$ is connected, non-complete graph of order $n$, then
  
  $1 \leq \kappa(G) \leq n - 2$
Connectivity properties (cont.)

- $\kappa(G) \geq 2 \iff G$ is connected and has no cut vertices
- A vertex lies on a cycle $\not\implies$ it is not a cut vertex
  - $\implies$ (Ex13, S1.1.2, H) Every vertex of a connected graph $G$ lies on at least one cycle $\not\implies \kappa(G) \geq 2$
  - (Ex14, S1.1.2, H) $\kappa(G) \geq 2$ implies $G$ has at least one cycle
- (Ex12, S1.1.2, H) $G$ has a cut vertex vs. $G$ has a bridge

**Proposition (1.2.14, W)**
An edge $e$ is a bridge $\iff$ $e$ lies on no cycle of $G$
- Or equivalently, an edge $e$ is not a bridge $\iff e$ lies on a cycle of $G$
Connectivity and minimal degree

• (Ex15, S1.1.2, H)
• \( \kappa(G) \leq \delta(G) \)
• If \( \delta(G) \geq n - 2 \), then \( \kappa(G) = \delta(G) \)
Edge-connectivity

• A proper subset $F \subset E$ is edge cut set if the graph $G - F$ is disconnected

• The edge-connectivity $\lambda(G)$ is the minimal size of edge cut set

• $\lambda(G) = 0$ if $G$ is disconnected

• Proposition (1.4.2, D) If $G$ is non-trivial, then $\kappa(G) \leq \lambda(G) \leq \delta(G)$
Average (minimal) degree implies connectivity

- **Theorem (1.4.3, D, Mader 1972)** Every graph $G$ with $d(G) \geq 4k$ has a $(k + 1)$-connected subgraph $H$ such that $d(H) > d(G) - 2k$. 
Bipartite graphs

• **Theorem** (1.2.18, W, Kőnig 1936)
  A graph is bipartite $\iff$ it contains no odd cycle

**Proposition** (1.2.15, W) Every closed odd walk contains an odd cycle
Complete graph is a union of bipartite graphs

- The union of graphs $G_1, \ldots, G_k$, written $G_1 \cup \cdots \cup G_k$, is the graph with vertex set $\bigcup_{i=1}^{k} V(G_i)$ and edge set $\bigcup_{i=1}^{k} E(G_i)$.

- Consider an air traffic system with $k$ airlines
  - Each pair of cities has direct service from at least one airline
  - No airline can schedule a cycle through an odd number of cities
  - Then, what is the maximum number of cities in the system?

- Theorem (1.2.23, W) The complete graph $K_n$ can be expressed as the union of $k$ bipartite graphs $\iff n \leq 2^k$
Bipartite subgraph is large

• **Theorem (1.3.19, W)** Every loopless graph $G$ has a bipartite subgraph with at least $|E|/2$ edges