Lecture 7: Coloring

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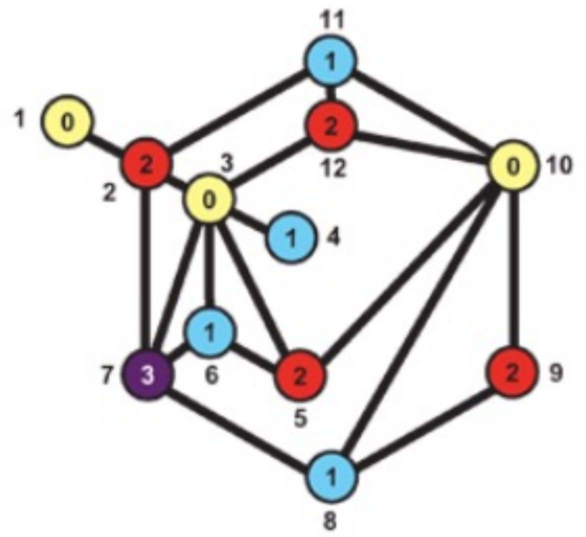
https://shuaili8.github.io/Teaching/CS445/index.html

Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \ldots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable
- In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable. G is said to be k-chromatic
- Examples

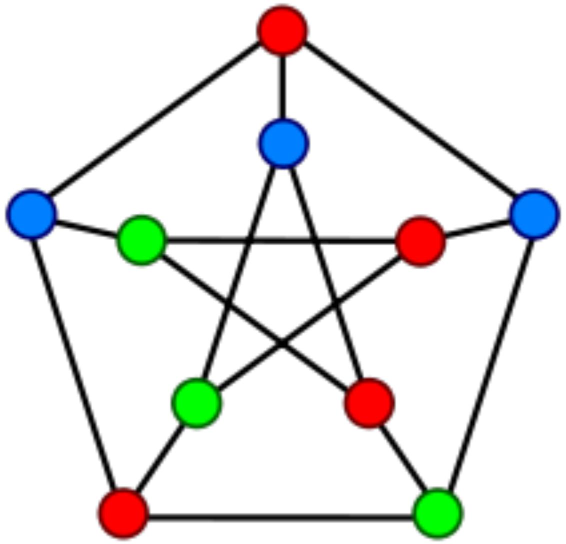
$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$

$$\chi(E_n) = 1,$$ ←Empty graph

$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite ⟺ it is 2-colorable

Theorem (1.2.18, W, Kőnig 1936)
A graph is bipartite ⟺ it contains no odd cycle

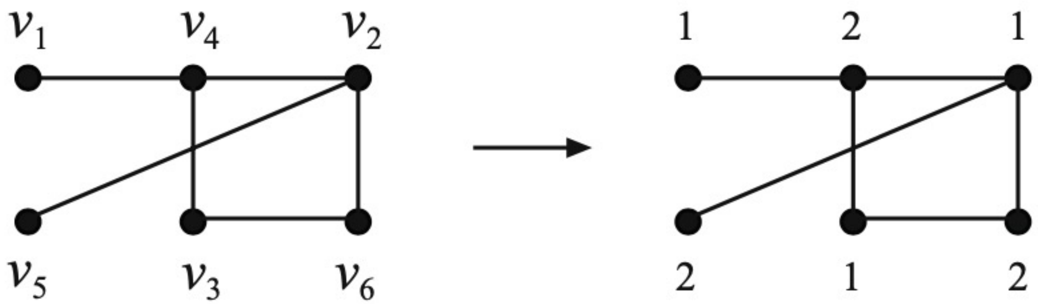
Bounds on Chromatic number

- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \dots, v_n$
- Next, order the available colors $(1,2,\dots,n)$ in some way
  - Start coloring by assigning color 1 to vertex v_1
  - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
  - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors

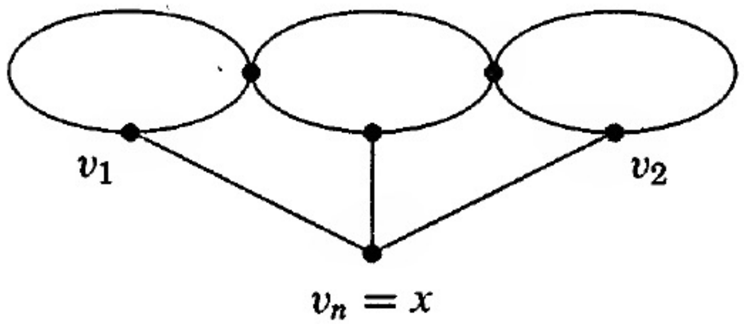


Bound using the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
  The equality is obtained for complete graphs and odd cycles

Brooks's theorem

- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$

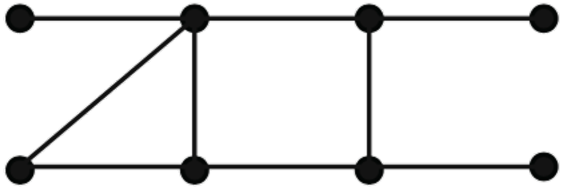


- ⇒The Petersen graph is 3-colorable

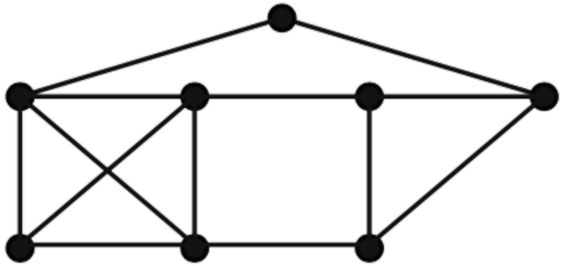
Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$

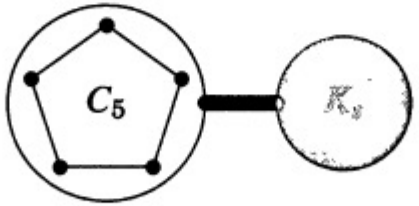
G_1



G_2



- Theorem (1.44, H; 5.1.7, W) For any graph G, $\chi(G) \geq \omega(G)$
- Example (5.1.8, W) For $G = C_{2r+1} \vee K_s$, $\chi(G) > \omega(G)$



Chromatic number and independence number

- Theorem (1.45, H; 5.1.7, W; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The independence number of a graph G, denoted as $\alpha(G)$, is the largest size of an independent set

In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow$ $V(G)$ is the union of k independent sets

Extremal properties for k-chromatic graphs

- Proposition (5.2.5, W) Every k-chromatic graph with n vertices has at least $\binom{k}{2}$ edges
  - Equality holds for a complete graph plus isolated vertices.

> In a proper coloring, each color class is an independent set. Then G is k-colorable $\Leftrightarrow V(G)$ is the union of k independent sets

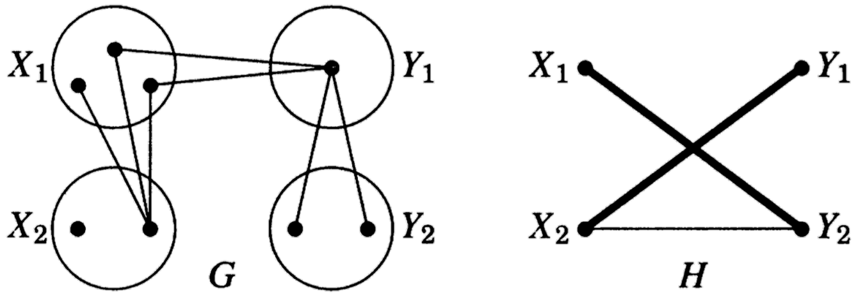
- The Turán graph $T_{n,r}$ is the complete r-partite graph with n vertices whose partite sets differ by at most 1 vertex
  - Every partite set has size $\lfloor n/r \rfloor$ or $\lceil n/r \rceil$
- Lemma (5.2.8, W) Among simple r-partite (that is, r-colorable) graphs with n vertices, the Turán graph is the unique graph with the most edges
- Turán's Theorem (5.2.9, W; Turán 1941) Among the n-vertex simple K_{r+1}-free graphs, $T_{n,r}$ has the maximum number of edges

Color-critical

- If $\chi(H) < \chi(G) = k$ for every proper subgraph H, then G is color-critical or k-critical
- K_2 is the only 2-critical graph
  K_1 is the only 1-critical graph
- (5.2.12, W) A graph with no isolated vertices is color-critical $\Leftrightarrow$ $\chi(G - e) < \chi(G)$ for every edge $e \in E(G)$
- Proposition (5.2.13, W) Let G be a k-critical graph
  (a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k - 1$ colors all appear on $N(v)$
  $\Rightarrow \delta(G) \geq k - 1$
  (b) For every $e \in E(G)$, every proper $(k - 1)$-coloring of $G - e$ gives the same color to the two endpoints of e

Color-critical has edge-connectivity

- Theorem (5.2.16, W; Dirac 1953) Every k-critical graph is $(k-1)$-edge-connected
- Lemma (5.2.15, W; Kainen) Let G be a graph with $\chi(G) > k$ and let X, Y be a partition of $V(G)$. If $G[X]$ and $G[Y]$ are k-colorable, then the edge cut $[X, Y]$ has at least k edges

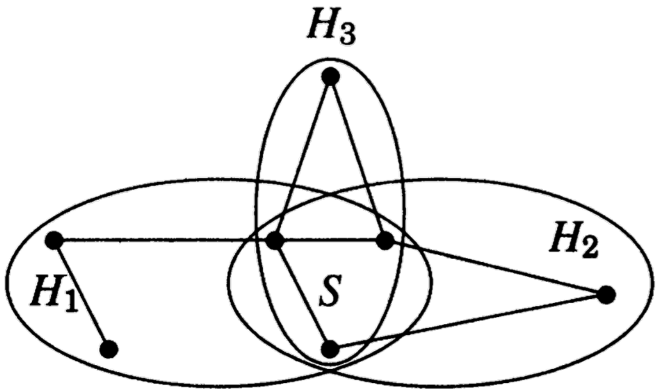


Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Remark (4.1.8, W) Every minimal disconnecting set of edges is an edge cut

Color-critical and vertex cut set

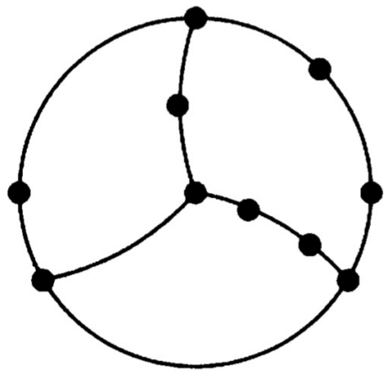
- Let S be a set of vertices in a graph G. An S-lobe of G is an induced subgraph of G whose vertex set consists of S and the vertices of a component in $G - S$



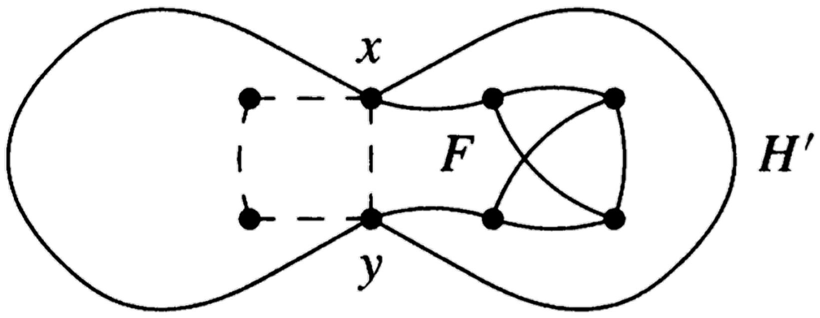
- Proposition (5.2.18, W) If G is k-critical, then G has no clique cutset. In particular, if G has a cutset $S = \{x, y\}$, then x, y are non-adjacent and G has an S-lobe H such that $\chi(H + xy) = k$

Chromatic number 4 has a K_4-subdivision

- Theorem (5.2.20, W; Dirac 1952) Every graph with chromatic number at least 4 contains a K_4-subdivision



a subdivision of K_4



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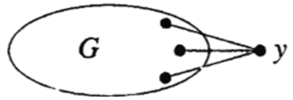
Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



Hajós' conjecture

- Hajós' conjecture [1961]: Every k-chromatic graph contains a subdivision of K_k
- $k = 2$: Every 2-chromatic graph has a nontrivial path
- $k = 3$: Every 3-chromatic graph has a cycle
- It is open for $k = 5,6$
- Exercise (Ex5.2.40, W) It is false for $k = 7$ or 8

Chromatic Polynomials

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $\chi(G; k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
  - How many different colorings of K_4 using 4 colors?
    - $4 \times 3 \times 2 \times 1$
    - $\chi(K_4; 4) = 24$
  - How many different colorings of K_4 using 6 colors?
    - $6 \times 5 \times 4 \times 3$
    - $\chi(K_4; 6) = 360$
  - How many different colorings of K_4 using 2 colors?
    - 0
    - $\chi(K_4; 2) = 0$

Examples

- If $k \geq n$

$$\chi(K_n; k) = k(k-1)\cdots(k-n+1)$$

- If $k < n$

$$\chi(K_n; k) = 0$$

- G is k-colorable $\Leftrightarrow$ $\chi(G) \leq k$ $\Leftrightarrow \chi(G; k) > 0$
- $\chi(G) = \min\{k \geq 1: \chi(G; k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

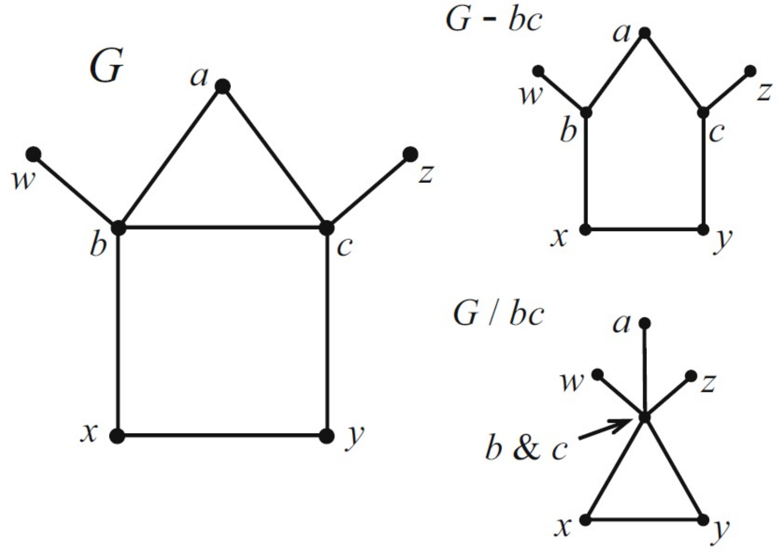


FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G. Then

$$\chi(G; k) = \chi(G - e; k) - \chi(G/e; k)$$

Use chromatic recurrence to compute $\chi(G; k)$

- Example: Compute $\chi(P_3; k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $\chi(P_3; 1) = 0, \chi(P_3; 2) = 2$

FIGURE 1.102. Two 2-colorings of P_3

- Example: What is $\chi(K_n - e; k)$?

More examples

- Path P_{n-1} has $n-1$ edges (n vertices)
$$\chi(P_{n-1}; k) = k(k-1)^{n-1}$$
- Any tree T on n vertices
$$\chi(T; k) = k(k-1)^{n-1}$$
- Cycle C_n
$$\chi(C_n; k) = (k-1)^n + (-1)^n(k-1)$$
  - When n is odd, $\chi(C_n; 2) = 0, \chi(C_n; 3) > 0$
  - When n is even, $\chi(C_n; 2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $\chi(G;k)$ is a polynomial in k of degree n
  - The leading coefficient of $\chi(G;k)$ is 1
  - The constant term of $\chi(G;k)$ is 0
    - If G has i components, then the coefficients of $k^0, \ldots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $\chi(G;k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow \chi(G;k) = k(k-1)^{n-1}$

      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n-1$ edges

    - A graph G is complete $\Leftrightarrow \chi(G;k) = k(k-1)\cdots(k-n+1)$

Simplicial elimination ordering

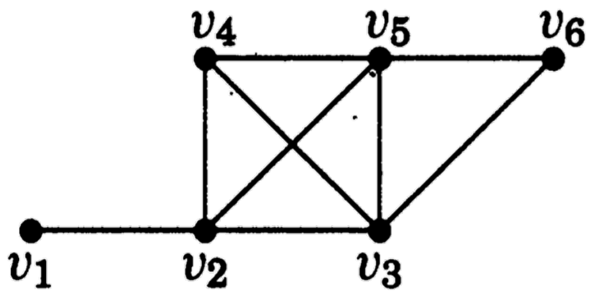
- Roots for the chromatic polynomials?
  Fundamental theorem of algebra
- A vertex of G is simplicial if its neighborhood in G induces a clique
- A simplicial elimination ordering is an ordering $v_n, \dots, v_1$ for deletion of vertices s.t. each vertex v_i is a simplicial vertex of the graph reduced by $\{v_1, \dots, v_i\}$
- Chromatic polynomials
  If we have colored $v_1, \dots, v_{i-1}$, then there are $k - d(i)$ ways to color v_i where $d(i) = |N(v_i) \cap \{v_1, \dots, v_{i-1}\}|$. Thus

$$\chi(G; k) = \prod_{i=1}^{n} (k - d(i))$$

Nice factorization property!

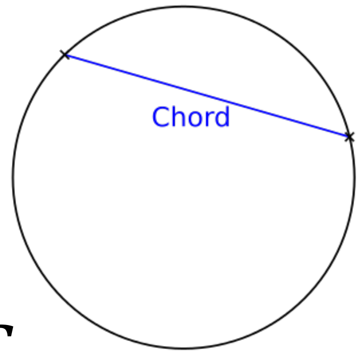
Examples

- In a tree, a simplicial elimination ordering is a successive deletion of leaves
  - Another proof for $\chi(T;k) = k(k-1)^{n-1}$
- Example (5.3.13, W) $v_6, \dots, v_1$ is a simplicial elimination ordering. The values $d(i)$ are $0,1,1,2,3,2$. Thus the chromatic polynomial is $k(k-1)(k-1)(k-2)(k-3)(k-2)$



- Exercise (Ex 5.3.19, W) There exists some graph without simplicial elimination ordering but has a nice factorization form for chromatic polynomial
  - The existence of simplicial elimination ordering is a sufficient condition for the chromatic polynomial having all real roots, but not necessary

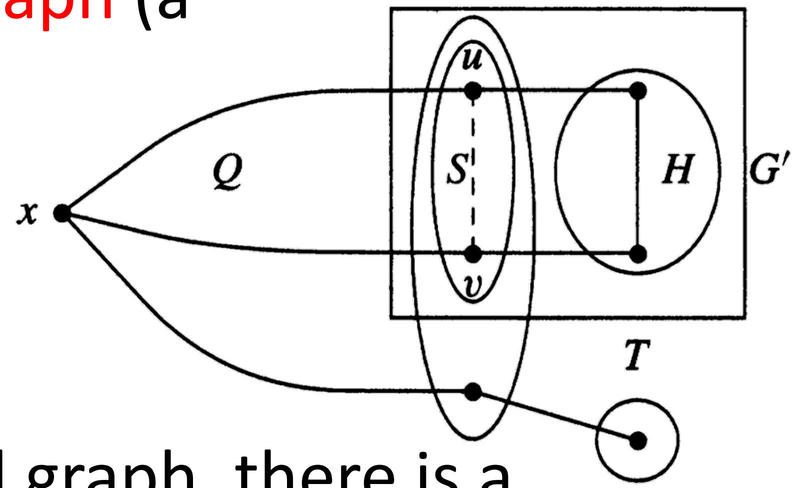
Chordal graphs



- A chord of a cycle C is an edge not in C whose endpoints lie in C
- A chordless cycle in G is a cycle of length at least 4 that has no chord
- Theorem (5.3.17, W; Dirac 1961) A simple graph has a simplicial elimination ordering $\Leftrightarrow$ it is a chordal graph (a simple graph without chordless cycle)
- TONCAS!
- Further

$$\chi(C_n; k) = (k-1)^n + (-1)^n (k-1)$$

does not have a degree-1 decomposition

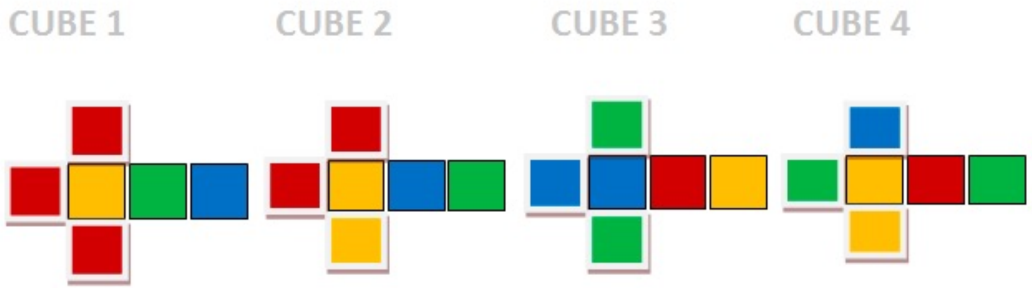


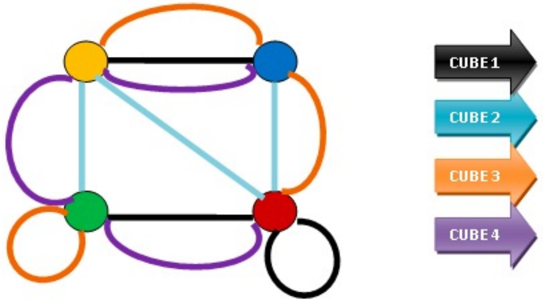
- Lemma (5.3.16, W) For every vertex x in a chordal graph, there is a simplicial vertex of G among the vertices farthest from x

Proof Using Coloring

Example -- Instant Insanity 四色方柱问题 (1.2, L)

- Problem make a stack of these cubes so that all four colors appear on each of the four sides of the stack
- An edge indicates that the two adjacent colors occur on opposite faces of the cube
- Problem necessary to find two subgraphs s.t.
  - are regular of degree 2
  - four edges, one from each cube
  - no edge in common for the two subgraphs





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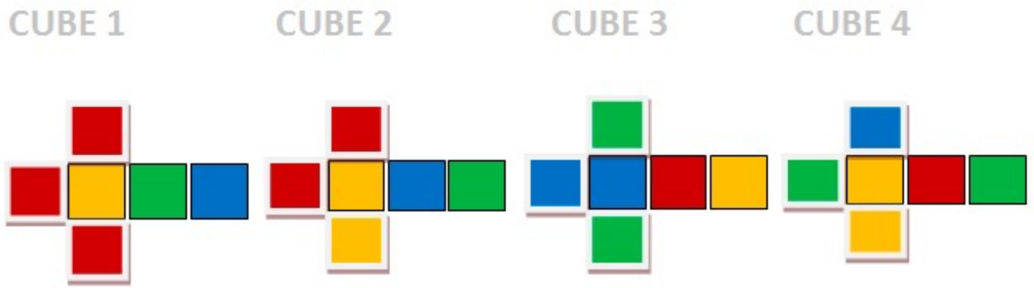
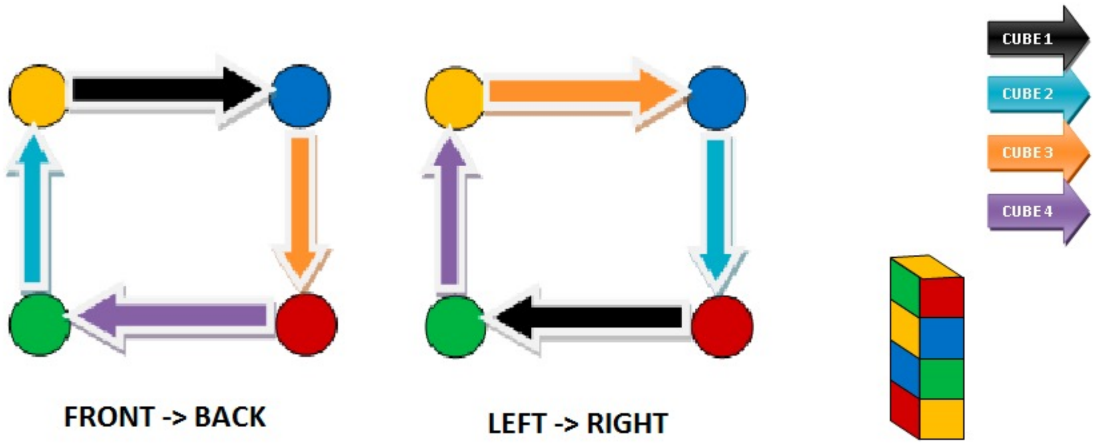
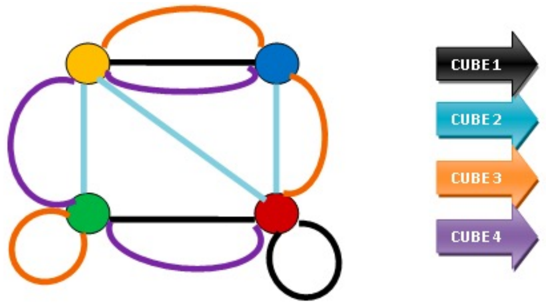
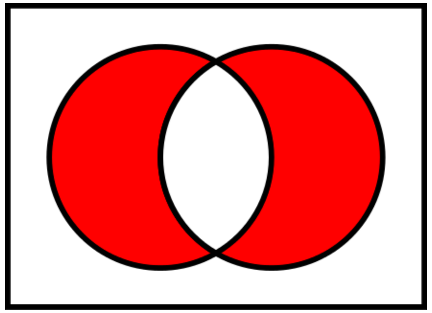






An example about sets (1E, L)

- Let $A_1, \ldots, A_n$ be n distinct subsets of the n-set $N := \{1, \ldots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \leq i \leq n$, are all distinct
- Proof Consider a graph with vertices $A_1, \ldots, A_n$.
  - An edge of `color' x between A_i and A_j iff $A_i \Delta A_j = \{x\}$
  - Then the problem is equivalent to find y s.t. no color y
  - Notice that a cycle in this graph must have even length and each color appears even times
  - Then we can remove an edge if there is an edge with same color
  - Thus the number of colors remain the same and no cycle exists
  - By tree property, the number of edges is at most $n - 1$



Summary

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- Coloring, proper coloring, chromatic #
- Brooks's theorem
- Chromatic # vs. clique/independence #
- Turán graph
- Color-critical, w/ vertex/edge-connectivity
- Chromatic number 4 has a K_4-subdivision
- Chromatic polynomials, chromatic recurrence, path/trees/cycles, properties
- Simplicial elimination ordering, chordal graph, TONCAS
- Examples of proof with coloring

Questions?