Lecture 9: Ramsey Theory

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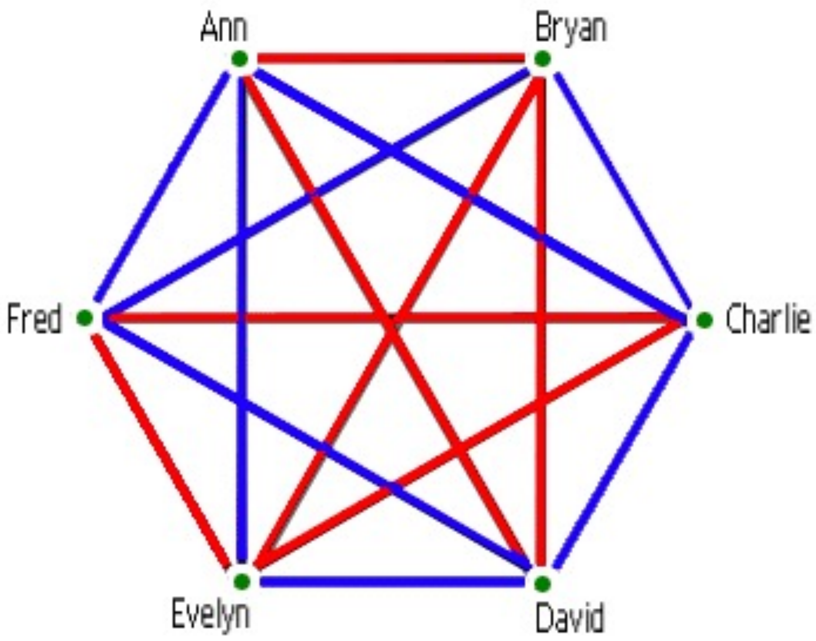
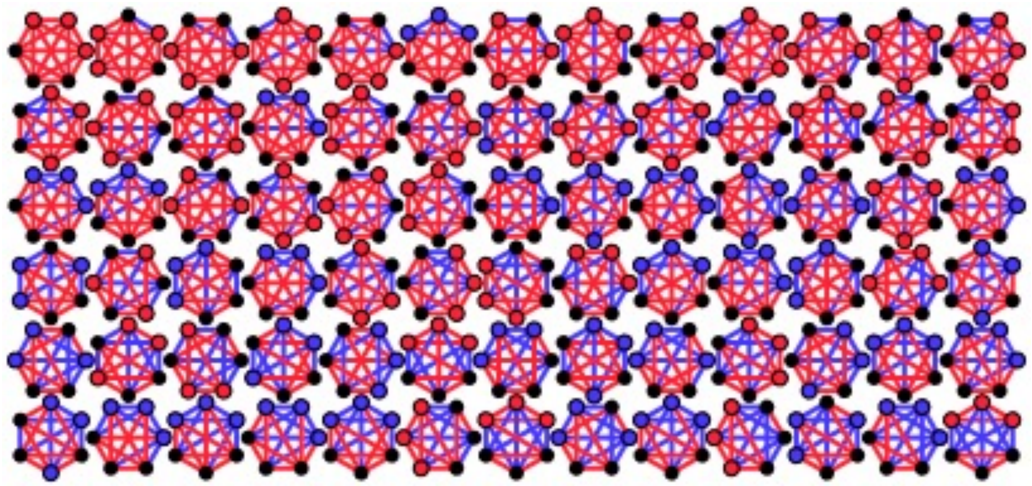
https://shuaili8.github.io

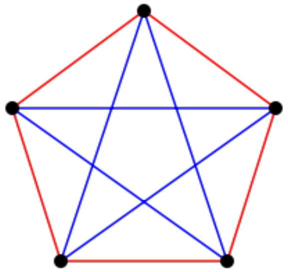
https://shuaili8.github.io/Teaching/CS445/index.html

The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?





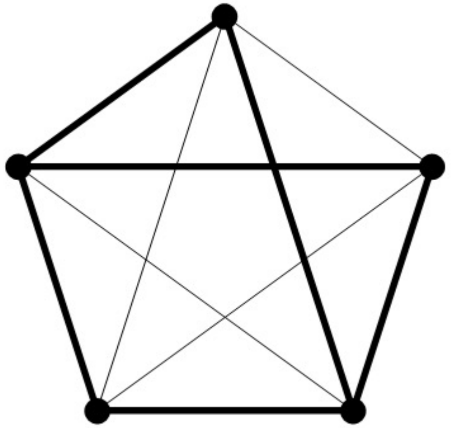


R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

https://plus.maths.org/content/friends-and-strangers
Wikipedia

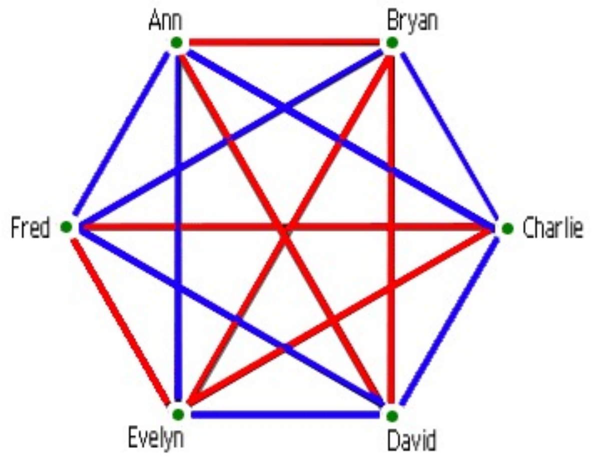
(classical) Ramsey number

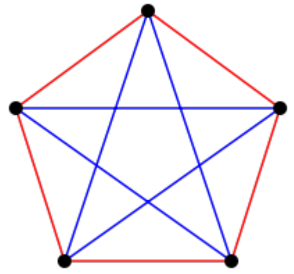
- A 2-coloring of the edges of a graph G is any assignment of one of two colors of each of the edges of G
- Let p and q be positive integers. The (classical) Ramsey number associated with these integers, denoted by $R(p,q)$, is defined to be the smallest integer n such that every 2-coloring of the edges of K_n either contains a red K_p or a blue K_q as a subgraph
- It is a typical problem of extremal graph theory



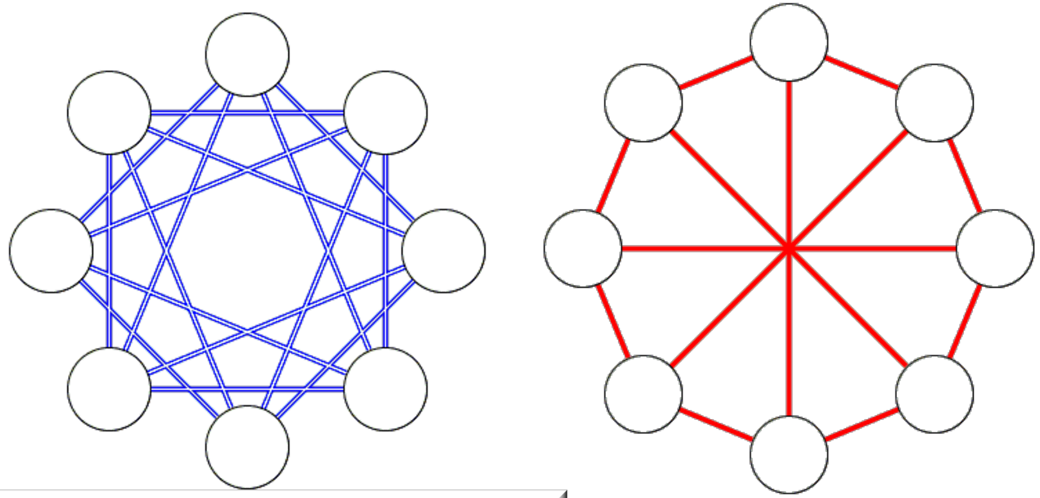
Examples

- $R(1,3) = 1$
- (Ex2, S1.8.1, H) $R(1,k) = 1$
- $R(2,4) = 4$
- (Ex3, S1.8.1, H) $R(2,k) = k$
- Theorem (1.61, H; 8.3.1, 8.3.9, W) $R(3,3) = 6$





Examples (cont.)



- **Theorem** (1.62, H; 8.3.10, W) $R(3,4) = 9$

> **Theorem** A finite graph G has an even number of vertices with odd degree

- (Ex4, S1.8.1, H) $R(p,q) = R(q,p)$

Values / known bounding ranges for Ramsey numbers $R(r, s)$ (sequence A212954 in the OEIS)

r \ s	1	2	3	4	5	6	7	8	9	10
1	**1**	1	1	1	1	1	1	1	1	1
2		**2**	3	4	5	6	7	8	9	10
3			**6**	9	14	18	23	28	36	40–42
4				**18**	25[10]	36–41	49–61	59[14]–84	73–115	92–149
5					**43–48**	58–87	80–143	101–216	133–316	149[14]–442
6						**102–165**	115[14]–298	134[14]–495	183–780	204–1171
7							**205–540**	217–1031	252–1713	292–2826
8								**282–1870**	329–3583	343–6090
9									**565–6588**	581–12677
10										**798–23556**

Bounds on Ramsey numbers

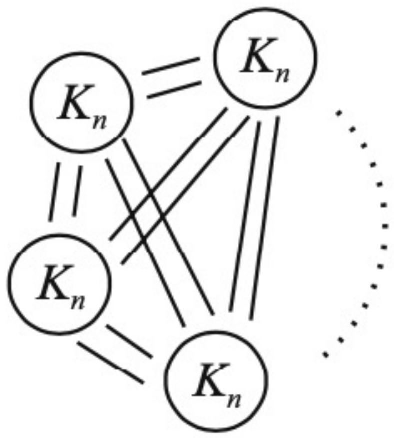
- Theorem (1.64, H; 2.28, H; 8.3.11, W) If $q \geq 2, q \geq 2$, then
$$R(p,q) \leq R(p-1,q) + R(p,q-1)$$
Furthermore, if both terms on the RHS are even, then the inequality is strict

  Theorem A finite graph G has an even number of vertices with odd degree

- Theorem (1.63, H; 2.29, H) $R(p,q) \leq \binom{p+q-2}{p-1}$
- Theorem (1.65, H) For integer $q \geq 3$, $R(3,q) \leq \frac{q^2+3}{2}$
- Theorem (1.66, H; 8.3.12, W; Erdős and Szekeres 1935)
If $p \geq 3$, $R(p,p) > \lfloor 2^{p/2} \rfloor$

Graph Ramsey Theory

- Given two graphs G and H, define the graph Ramsey number $R(G,H)$ to be the smallest value of n such that any 2-coloring of the edges of K_n contains either a red copy of G or a blue copy of H
  - The classical Ramsey number $R(p,q)$ would in this context be written as $R(K_p, K_q)$
- Theorem (1.67, H) If G is a graph of order p and H is a graph of order q, then $R(G,H) \leq R(p,q)$
- Theorem (1.68, H) Suppose the order of the largest component of H is denoted as $C(H)$.
  Then $R(G,H) \geq (\chi(G)-1)(C(H)-1)+1$



Graph Ramsey Theory (cont.)

- Theorem (1.69, H; 8.3.14, W) $R(T_m, K_n) = (m-1)(n-1)+1$

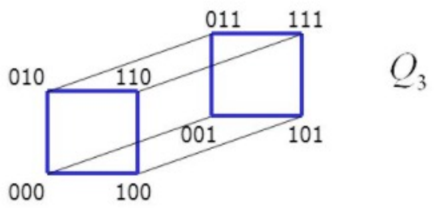
Theorem (1.45, H; Ex6, S1.6.2, H) For any graph G of order n,

$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n+1-\alpha(G)$$

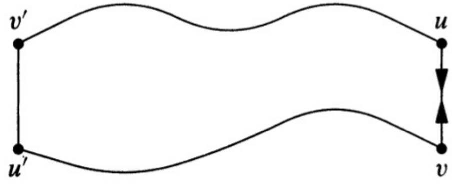
Proposition (5.2.13, W) Let G be a k-critical graph
(a) For every $v \in V(G)$, there is a proper coloring such that v has a unique color and other $k-1$ colors all appear on $N(v)$
$\Rightarrow \delta(G) \geq k-1$

Theorem (1.16, H) Let T be a tree of order $k+1$ with k edges. Let G be a graph with $\delta(G) \geq k$. Then G contains T as a subgraph

More on pigeonhole principle



- Proposition (8.3.1, W) Among 6 people, it is possible to find 3 mutual acquaintances or 3 mutual non-acquaintances
  - $\Leftrightarrow$ For every simple graph with 6 vertices, there is a triangle in G or in $\bar{G}$
- Theorem (8.3.2, W) If T is a spanning tree of the k-dimensional cube Q_k, then there is an edge of Q_k outside T whose addition to T creates a cycle of length at least $2k$



T is a tree of order n
$\Leftrightarrow$ Any two vertices of T are linked by a unique path in T

  - $\Rightarrow$ Every spanning tree of Q_k has diameter at least $2k-1$

More on pigeonhole principle 2

- Theorem (8.3.3, W; Erdős–Szekeres 1935) Every list of $\geq n^2 + 1$ distinct numbers has a monotone sublist of length $\geq n + 1$
  - Generalization. $(r-1)(s-1)+1$
- Theorem (8.3.4, W; Graham-Kleitman 1973) In every labeling of $E(K_n)$ using distinct integers, there is a walk of length at least $n - 1$ along which the labels strictly increase

Summary

- Ramsey number
- Graph Ramsey Theory
- More on pigeonhole principle

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Questions?