Lecture: More on Connectivity (2)

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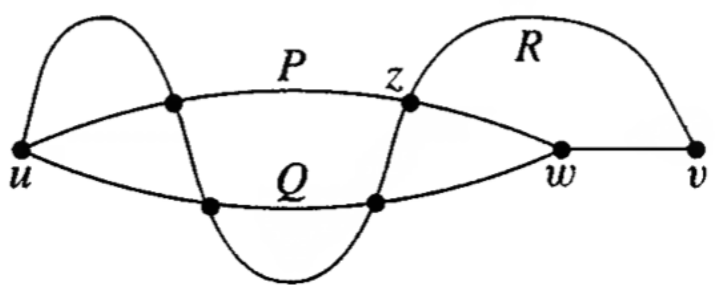
https://shuaili8.github.io/Teaching/CS445/index.html

2-Connected Graphs

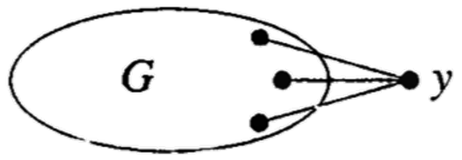
2-connected graphs

- Two paths from u to v are internally disjoint if they have no common internal vertex
- Theorem (4.2.2, W; Whitney 1932)
  A graph G having at least three vertices is 2-connected $\Longleftrightarrow$ for each pair $u, v \in V(G)$ there exist internally disjoint u, v-paths in G

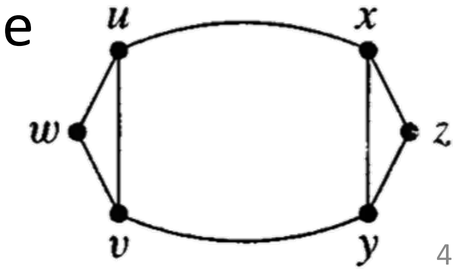


Equivalent definitions for 2-connected graphs

- Lemma (4.2.3, W; Expansion Lemma) If G is a k-connected graph, and G' is obtained from G by adding a new vertex y with at least k neighbors in G, then G' is k-connected



- Theorem (4.2.4, W) For a graph G with at least three vertices, TFAE
  - G is connected and has no cut-vertex
  - For all $x, y \in V(G)$, there are internally disjoint x, y-paths
  - For all $x, y \in V(G)$, there is a cycle through x and y
  - $\delta(G) \geq 1$ and every pair of edges in G lies on a common cycle



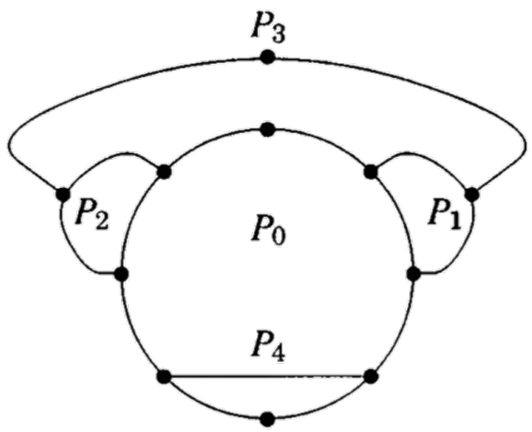
Subdivision keeps 2-connectivity

A subdivision of an edge e in G is a substitution of a path for e

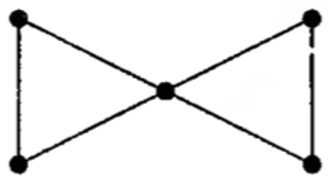
- Corollary (4.2.6, W) If G is 2-connected, then the graph G' obtained by subdividing an edge of G is 2-connected

Ear decomposition

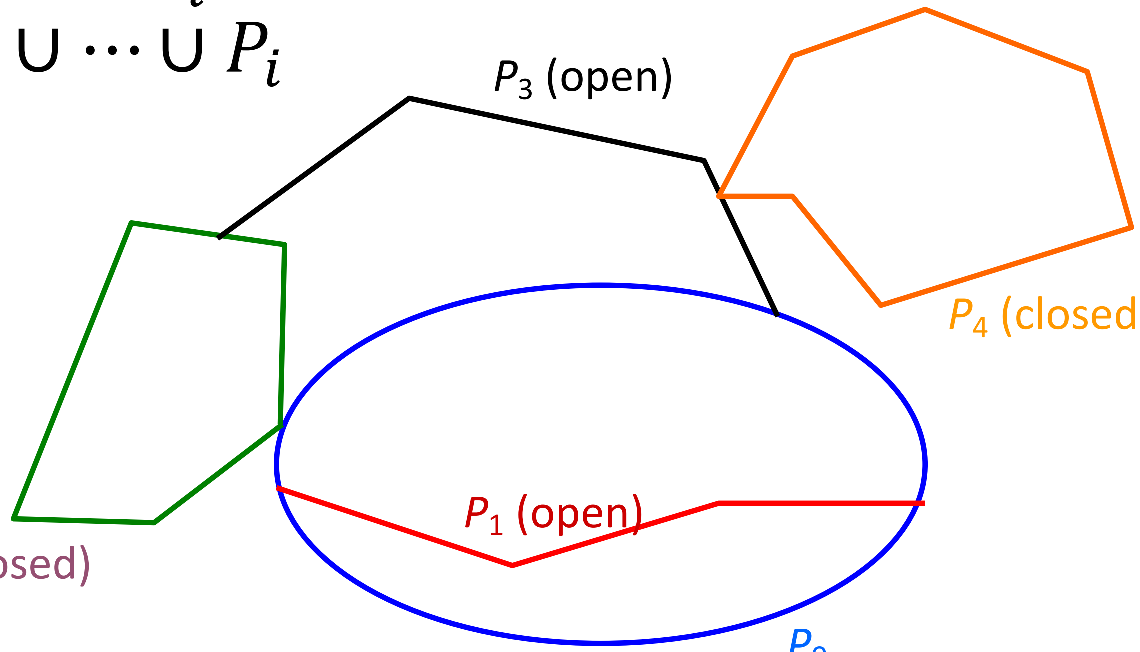
- An ear of a graph G is a maximal path whose internal vertices have degree 2 in G
- An ear decomposition of G is a decomposition $P_0, \dots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an ear of $P_0 \cup \cdots \cup P_i$
- Theorem (4.2.8, W)
  A graph is 2-connected $\Leftrightarrow$ it has an ear decomposition.
  Furthermore, every cycle in a 2-connected graph is the initial cycle in some ear decomposition



Closed-ear



- A closed ear of a graph G is a cycle C such that all vertices of C except one have degree 2 in G
- A closed-ear decomposition of G is a decomposition $P_0, \ldots, P_k$ such that P_0 is a cycle and P_i for $i \geq 1$ is an (open) ear or a closed ear in $P_0 \cup \cdots \cup P_i$



Closed-ear decomposition

- Theorem (4.2.10, W)
  A graph is 2-edge-connected $\Leftrightarrow$ it has a closed-ear decomposition.
  Every cycle in a 2-edge-connected graph is the initial cycle in some such decomposition

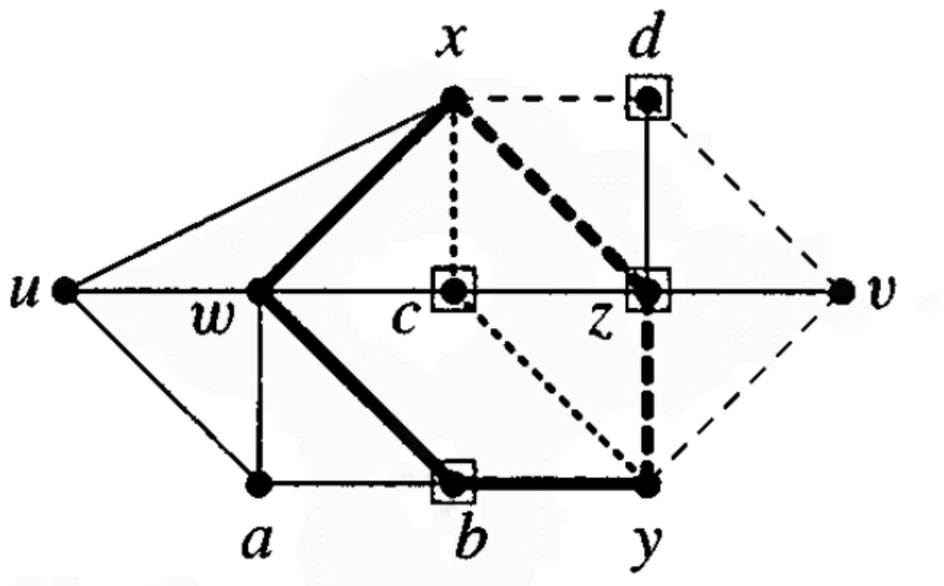
k-Connected and k-Edge-Connected graphs

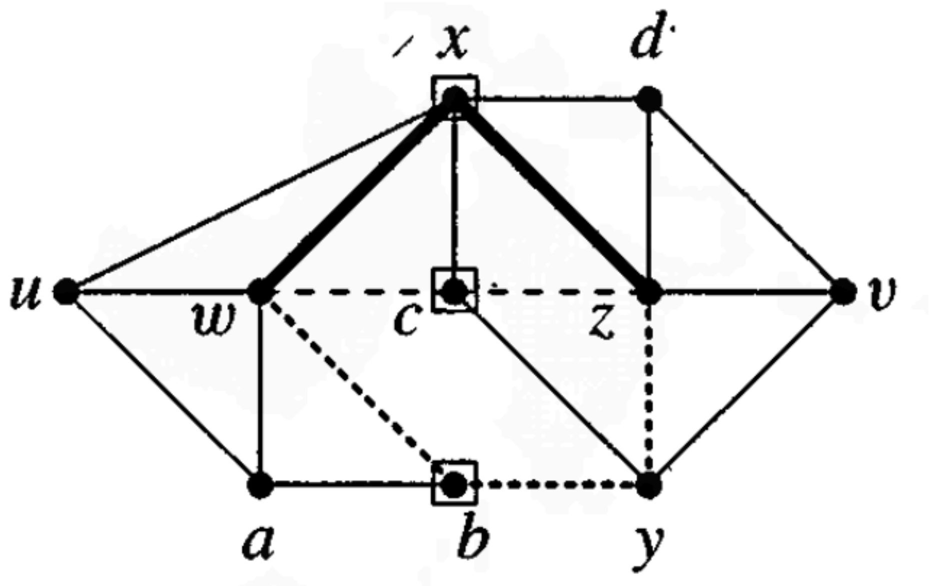
x, y-cut

- Given $x, y \in V(G)$, a set $S \subseteq V(G) - \{x, y\}$ is an x, y-separator or x, y-cut if $G - S$ has no x, y-path
  - Let $\kappa(x, y)$ be the minimum size of an x, y-cut
  - Let $\lambda(x, y)$ be the maximum size of a set of pairwise internally disjoint x, y-paths
  - $\kappa(x, y) \geq \lambda(x, y)$
- For $X, Y \subseteq V(G)$, an X, Y-path is a path having first vertex in X, last vertex in Y, and no other vertex in $X \cup Y$

Example (4.2.16, W)

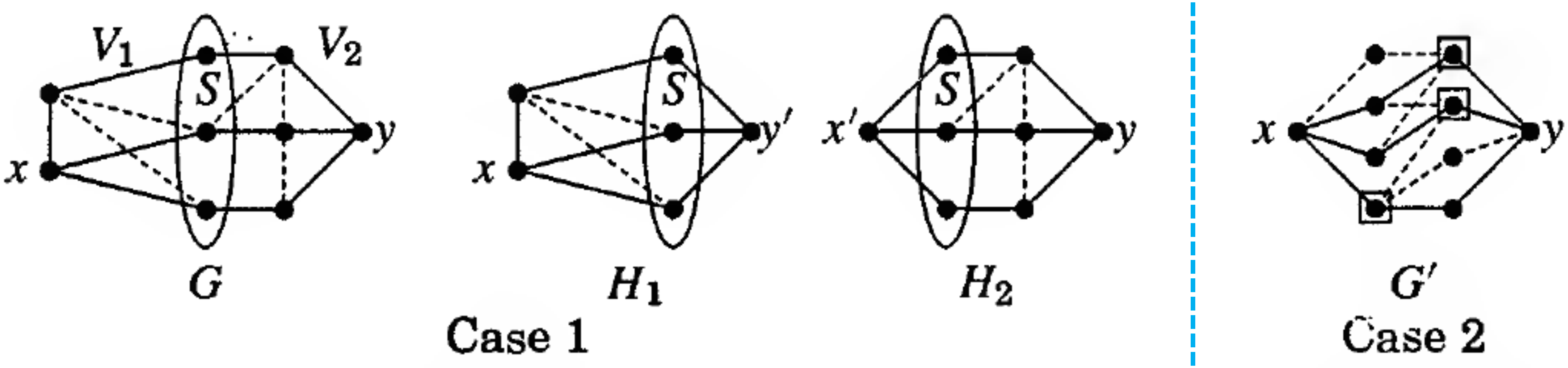
- $S = \{b, c, z, d\}$
- $\kappa(x, y) = \lambda(x, y) = 4$
- $\kappa(w, z) = \lambda(w, z) = 3$





Menger's Theorem

- Theorem (4.2.17, W; Menger, 1927) If x, y are vertices of a graph G and $xy \notin E(G)$, then $\kappa(x, y) = \lambda(x, y)$



Theorem (3.1.16, W; 1.53, H; 2.1.1, D; König 1931; Egeváry 1931) Let G be a bipartite graph. The maximum size of a matching in G is equal to the minimum size of a vertex cover of its edges

Edge version

- Theorem (4.2.19, W) If x and y are distinct vertices of a graph or digraph G, then the minimum size $\kappa'(x, y)$ of an x, y-disconnecting set of edges equals the maximum number $\lambda'(x, y)$ of pairwise edge-disjoint x, y-paths

Back to connectivity

- Theorem (4.2.21, W)

$$\kappa(G) = \min_{x,y \in V(G)} \lambda(x,y), \qquad \lambda(G) = \min_{x,y \in V(G)} \lambda'(x,y)$$

Application of Menger’s Theorem

CSDR

- Let $\boldsymbol{A} = A_1, \dots, A_m$ and $\boldsymbol{B} = B_1, \dots, B_m$ be two family of sets. A common system of distinct representatives (CSDR) is a set of m elements that is both an system of distinct representatives (SDR) for $\boldsymbol{A}$ and an SDR for $\boldsymbol{B}$

- Given some family of sets X, a system of distinct representatives for the sets in X is a 'representative' collection of distinct elements from the sets of X

$$
\begin{aligned}
S_1 &= \{2, 8\}, \\
S_2 &= \{8\}, \\
S_3 &= \{5, 7\}, \\
S_4 &= \{2, 4, 8\}, \\
S_5 &= \{2, 4\}.
\end{aligned}
$$

The family $X_1 = \{S_1, S_2, S_3, S_4\}$ does have an SDR, namely $\{2, 8, 7, 4\}$. The family $X_2 = \{S_1, S_2, S_4, S_5\}$ does not have an SDR.

- Theorem(1.52, H) Let $S_1, S_2, \dots, S_k$ be a collection of finite, nonempty sets. This collection has SDR $\Leftrightarrow$ for every $t \in [k]$, the union of any t of these sets contains at least t elements

Equivalent condition for CSDR

- Theorem (4.2.25, W; Ford-Fulkerson 1958) Families $\boldsymbol{A} = \{A_1, \ldots, A_m\}$ and $\boldsymbol{B} = \{B_1, \ldots, B_m\}$ have a common system of distinct representatives (CSDR) $\Leftrightarrow$

$$\left|\left(\bigcup_{i\in I} A_i\right) \cap \left(\bigcup_{j\in J} B_j\right)\right| \geq |I| + |J| - m$$

for every pair $I, J \subseteq [m]$