CS 445: Combinatorics

Shuai Li

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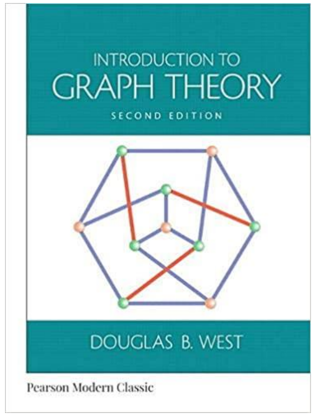
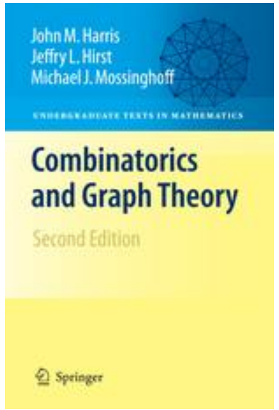
https://shuaili8.github.io

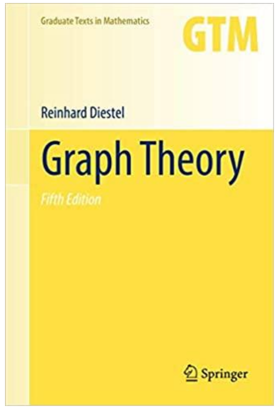
https://shuaili8.github.io/Teaching/CS445/index.html

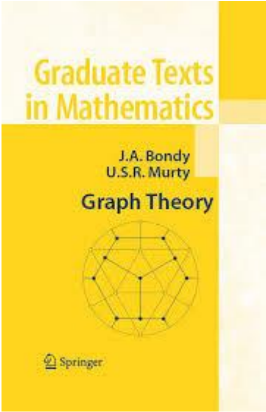
Textbooks & References

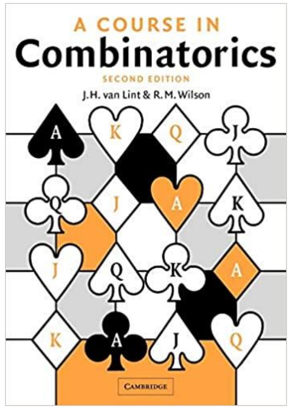
- Textbook:
  - Graph Theory, Reinhard Diestel
- References:
  - Introduction to Graph Theory, by Douglas West
  - Combinatorics and Graph Theory, by Harris, Hirst and Mossinghoff
  - Graph Theory, by Bondy and Murty
  - A Course in Combinatorics, J. H. Van Lint











Previous courses

- Discrete Mathematics
  - Basic concepts for graph theory
- Mathematical Foundations of Computer Science (CS 499)
  - Basic notions and hand shaking lemma
  - Graph isomorphism and graph score
  - Applications of handshake lemma: Parity argument
  - The number of spanning trees
  - Isomorphism of trees
  - Random graphs

Goal

- Knowledge of the basic problems for graph theory
  - Bipartite graphs/Matching/Coloring/Flows/…
- Knowledge of the important counting related results on graphs
- Familiar with the common proof techniques
- Awareness of the popular applications of graphs in many fields

Grading policy

- Attendance and participance: 5%
- Assignments: 35%
- Midterm exam: 20%
- Project: 10%
- Final exam: 30%

Honor code

- Discussions are encouraged
- Independently write-up homework and project
- Same reports and homework will be reported

Teaching Assistant

- Yueran Yang (杨悦然)
  - Email: yangyr99@sjtu.edu.cn
  - Senior undergraduate student majored in Mathematics
  - Research on recommendation systems and bioinformatics
  - Office hour: Friday 1-3 pm

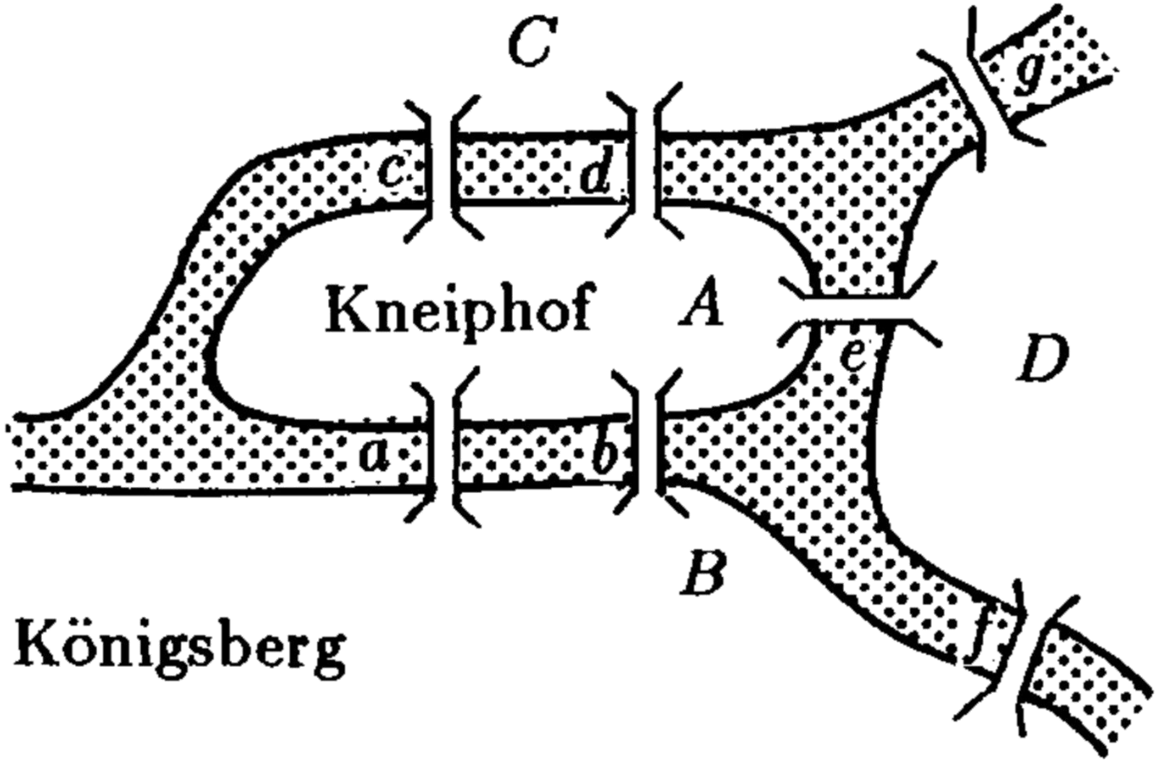
Course Outline

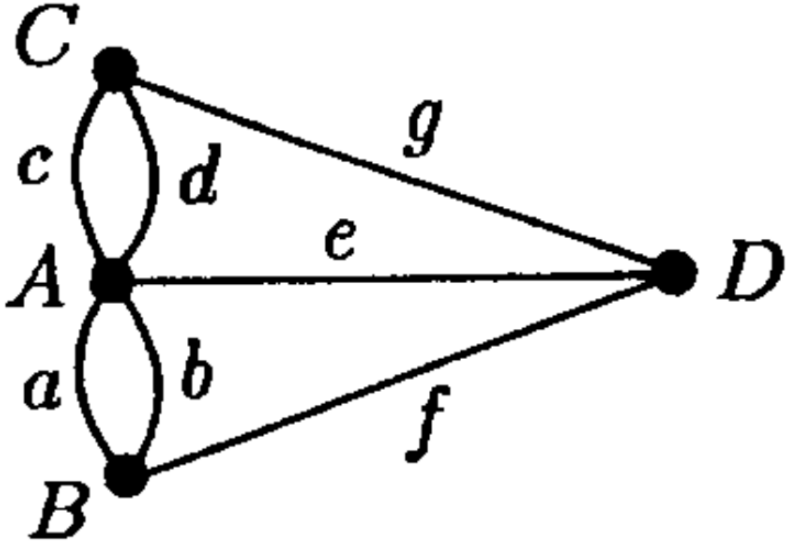
- Basics
    - Graphs, paths and cycles, connectivity, trees, bipartite graphs
- Matching, Covering and Packing
- Connectivity
- Planar Graphs
- Coloring
- Flows
- Ramsey theory

Introduction

Seven bridges of Königsberg 七桥问题

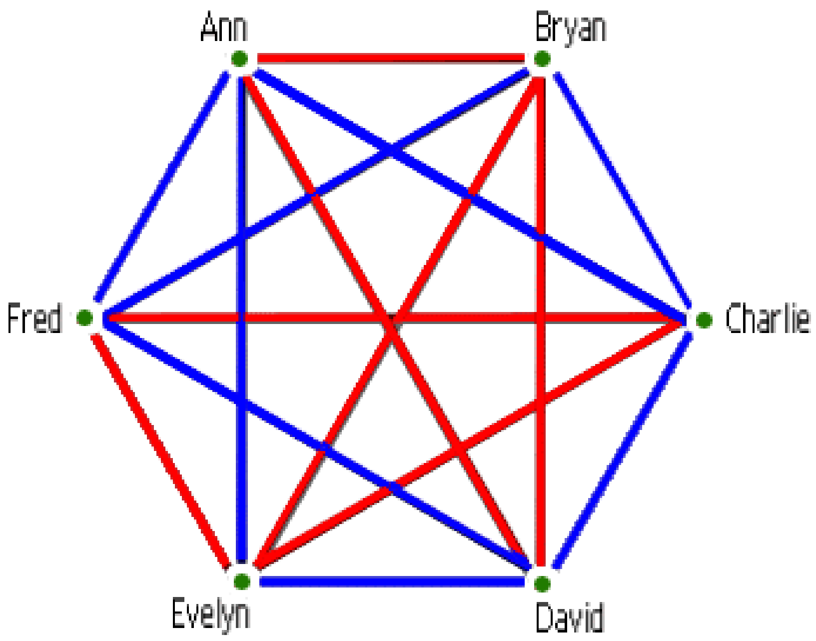
- Leonhard Euler 1736: Is it possible to make a walk through the city, returning to the starting point and crossing each bridge exactly once?

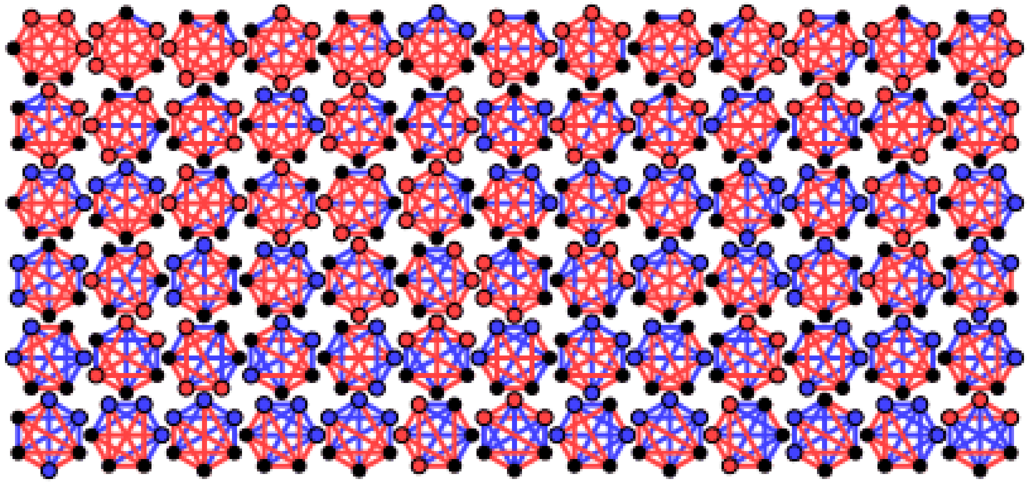


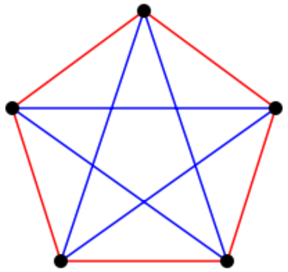


The friendship riddle

- Does every set of six people contain three mutual acquaintances or three mutual strangers?







R(3,3)=6
R(3,4)=R(4,3)=9
R(3,5)=R(5,3)=14
R(3,6)=R(6,3)=18

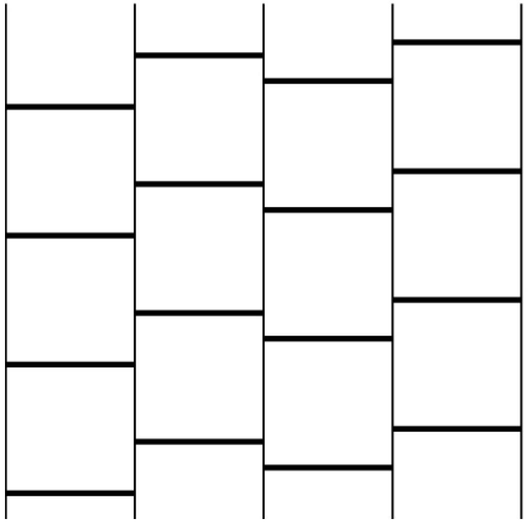
https://plus.maths.org/content/friends-and-strangers
Wikipedia

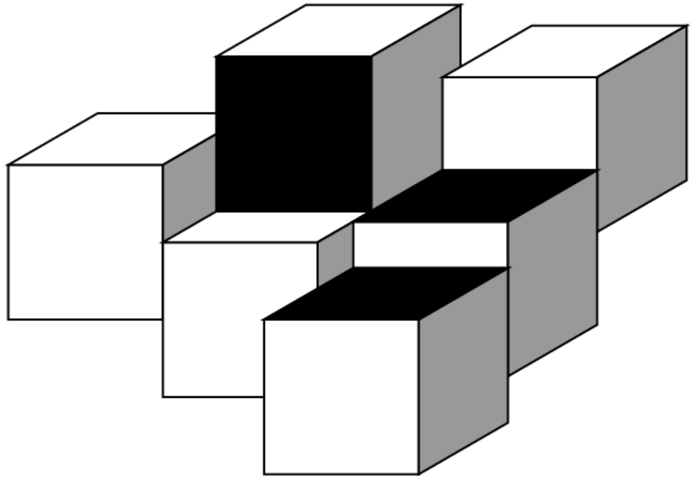
Examples of general combinatorics problems using graph theory

- Instant Insanity 四色方柱问题
  - make a stack of these cubes so that all four colors appear on each of the four sides of the stack

- A set problem
  - Let $A_1, \ldots, A_n$ be n distinct subsets of the n-set $N := \{1, \ldots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \leq i \leq n$, are all distinct

Keller’s conjecture



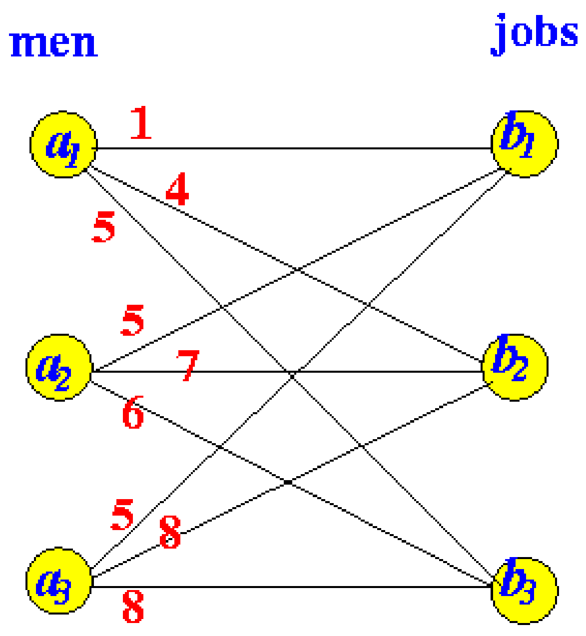


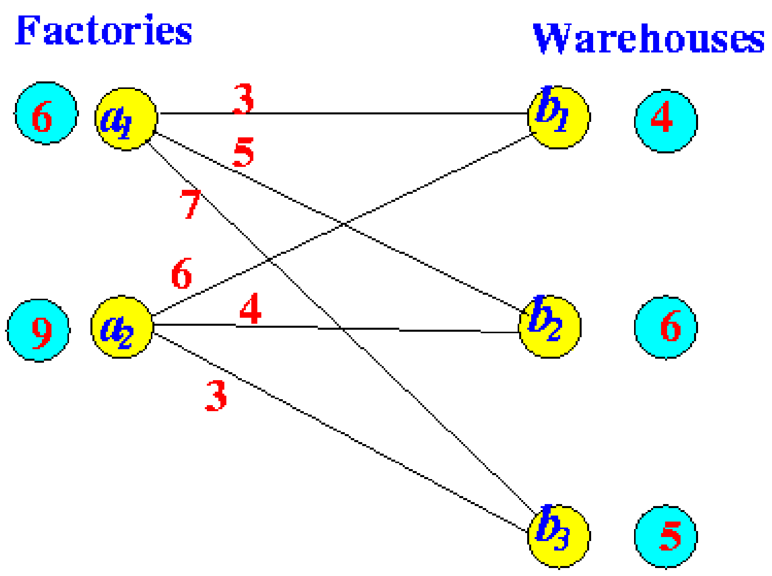
- In 1930, Keller conjectured that any tiling of n-dimensional space by translates of the unit cube must contain a pair of cubes that share a complete $(n-1)$-dimensional face

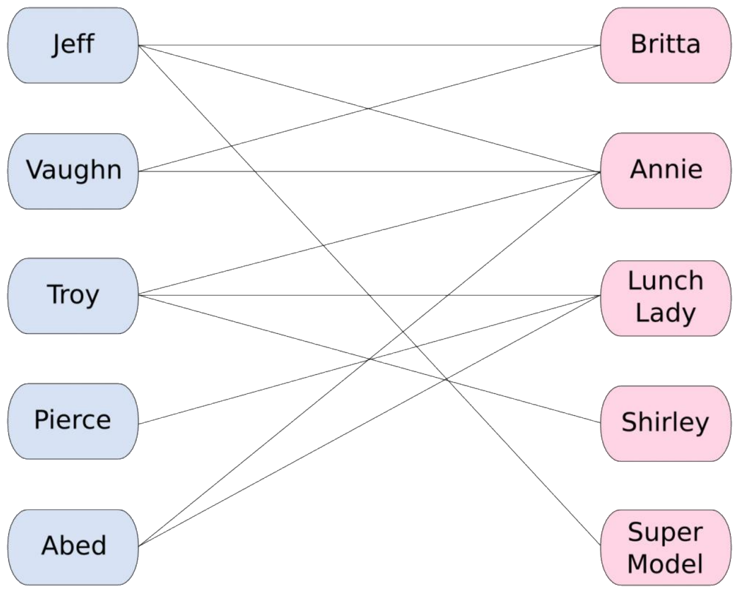
- Corrádi and Szabó transfer it into a graph theory problem
  - Constructing Keller graph
- The conjecture is solved by computer search recently (June 2020)

Reference: https://mp.weixin.qq.com/s/rTxW6MzSHuWUGC3vTIDbJw

Assignment problems

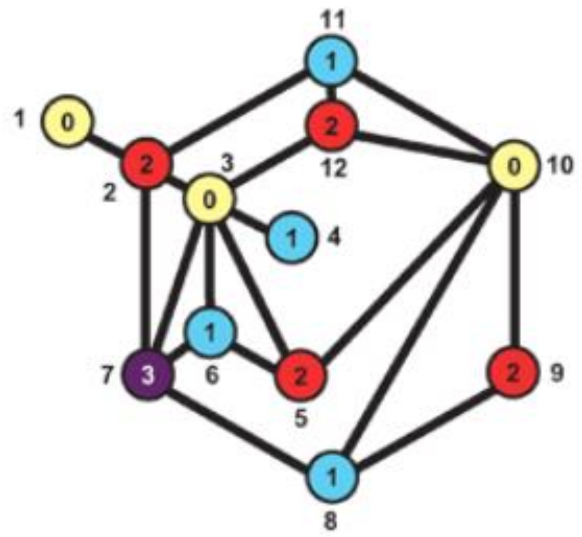






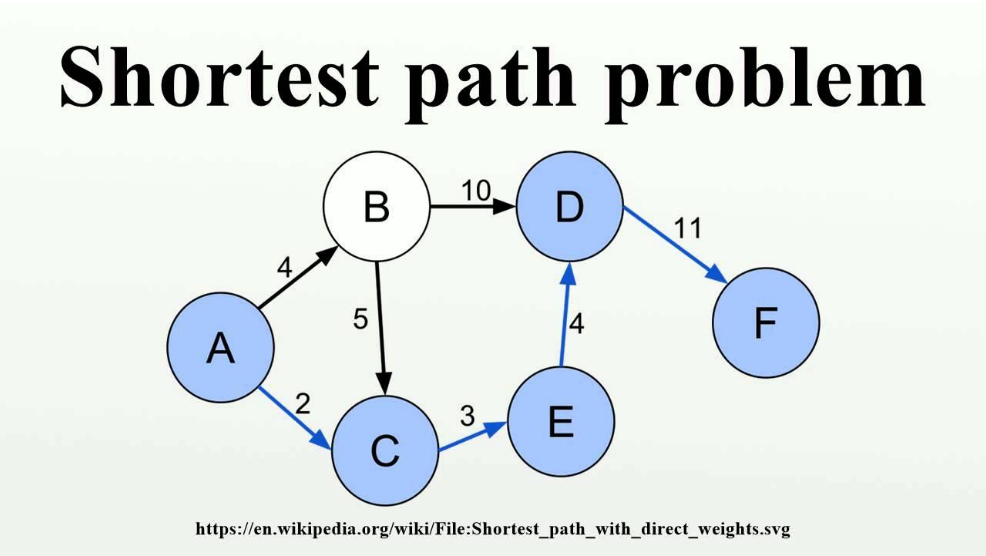
Scheduling and coloring

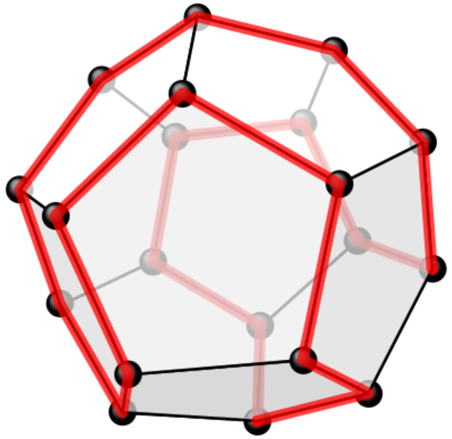
- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member

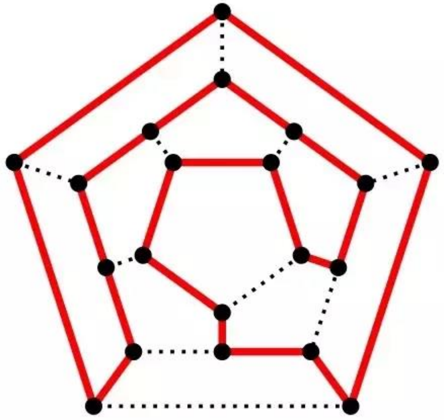


Routes in road networks

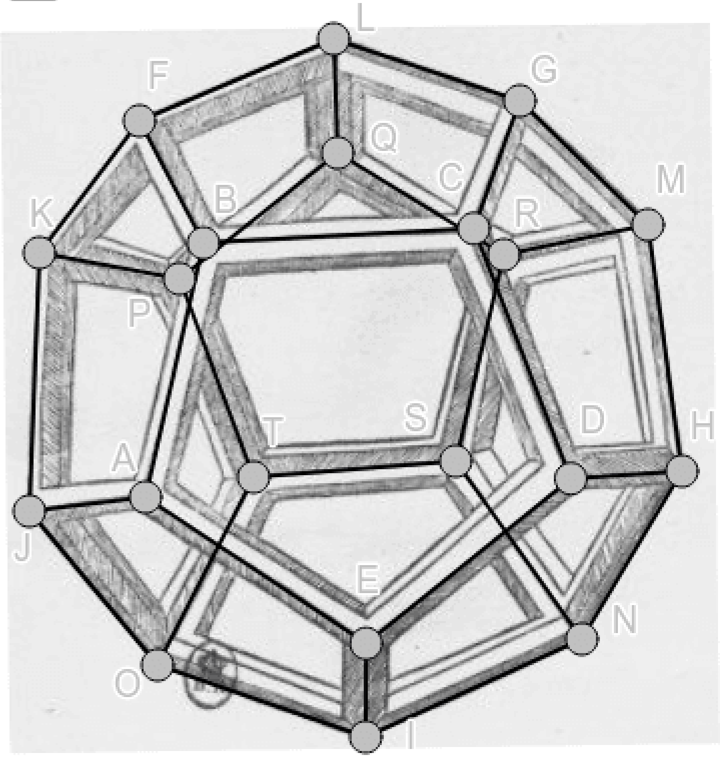
- How can we find the shortest route from x to y?
- If the vertices of the graph represent our house and other places to visit, then we may want to follow a route that visits every vertex exactly once, so as to visit everyone without overstaying our welcome
    - Hamilton circuit

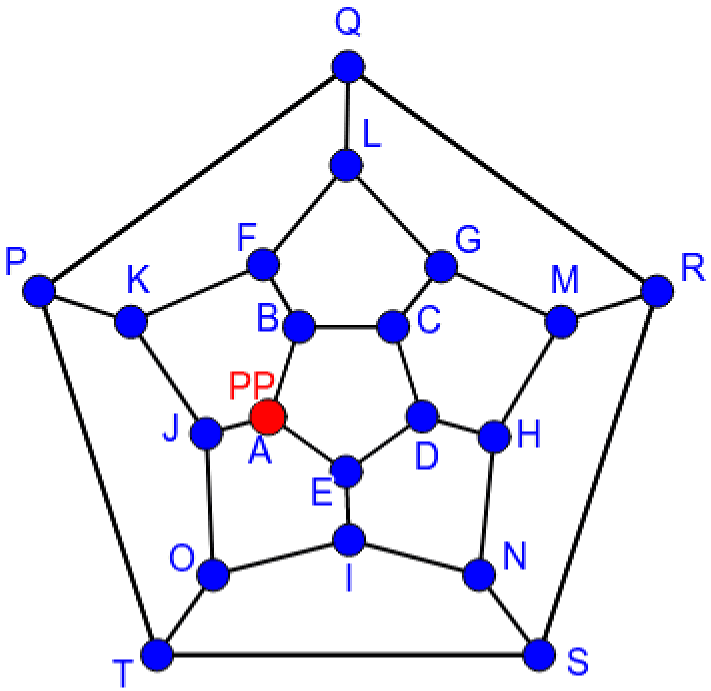






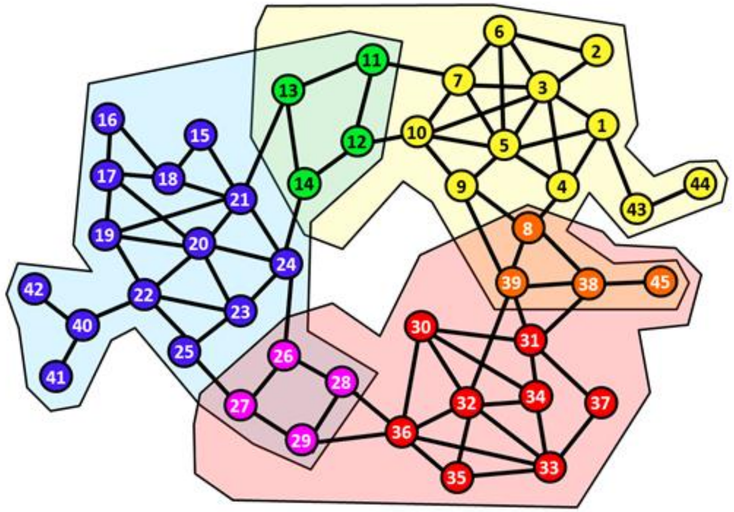
Leonardo da Vinci: DVODECEDRON

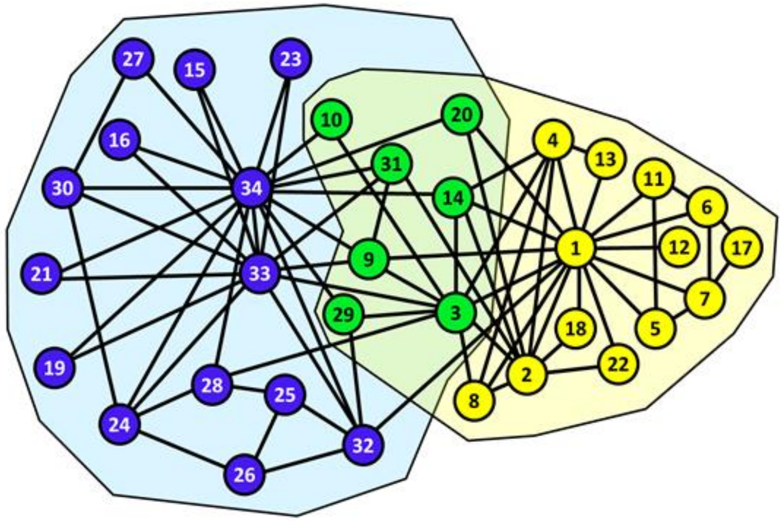




Social network

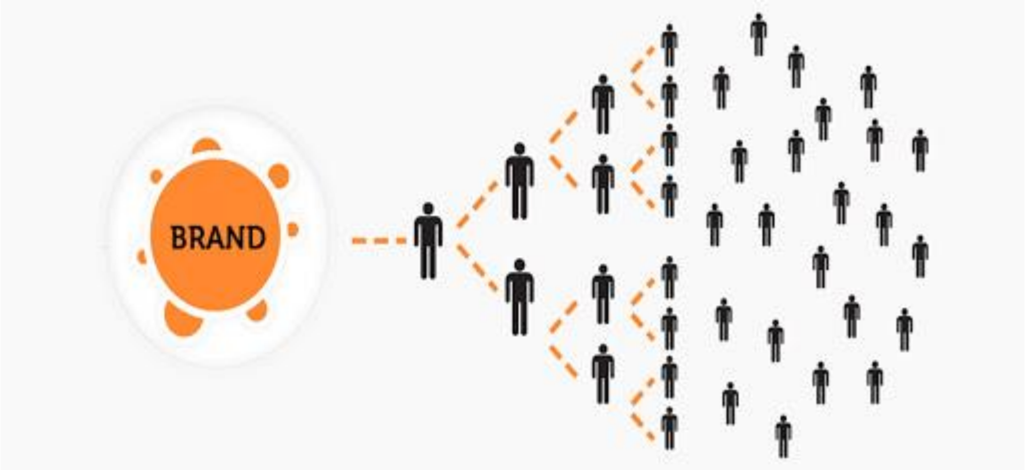
- Recommendation
- Clustering

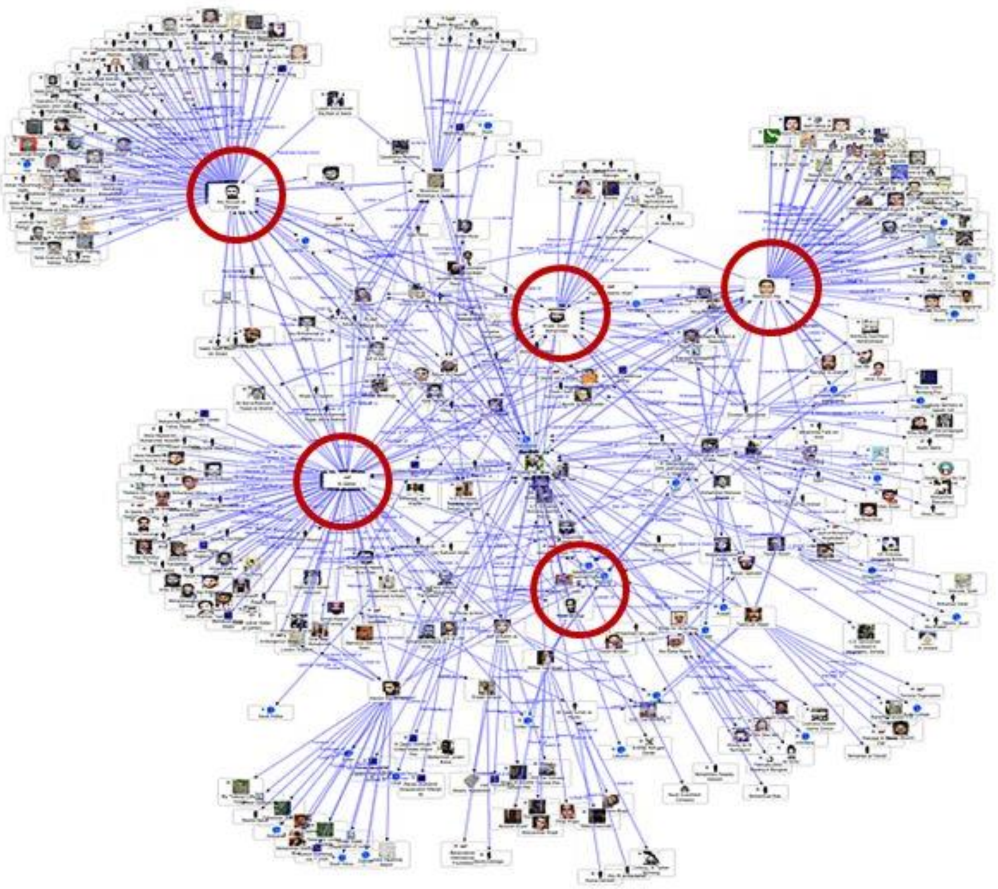




Influence maximization

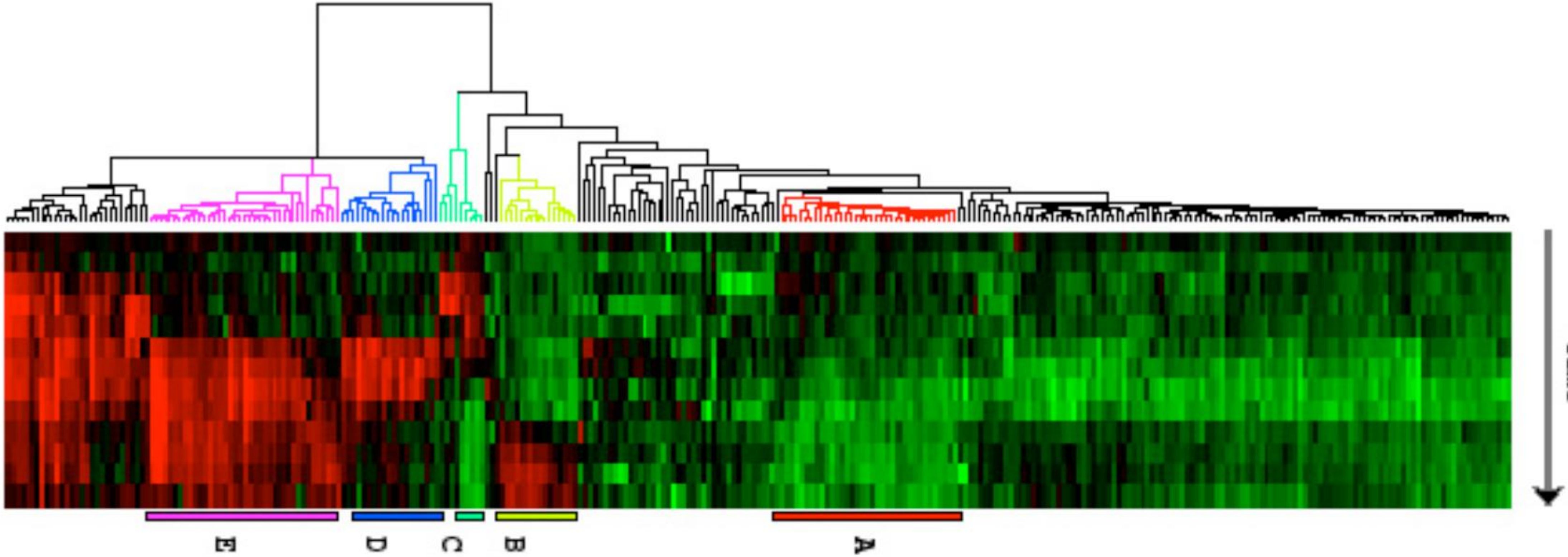
- Select the best seed set



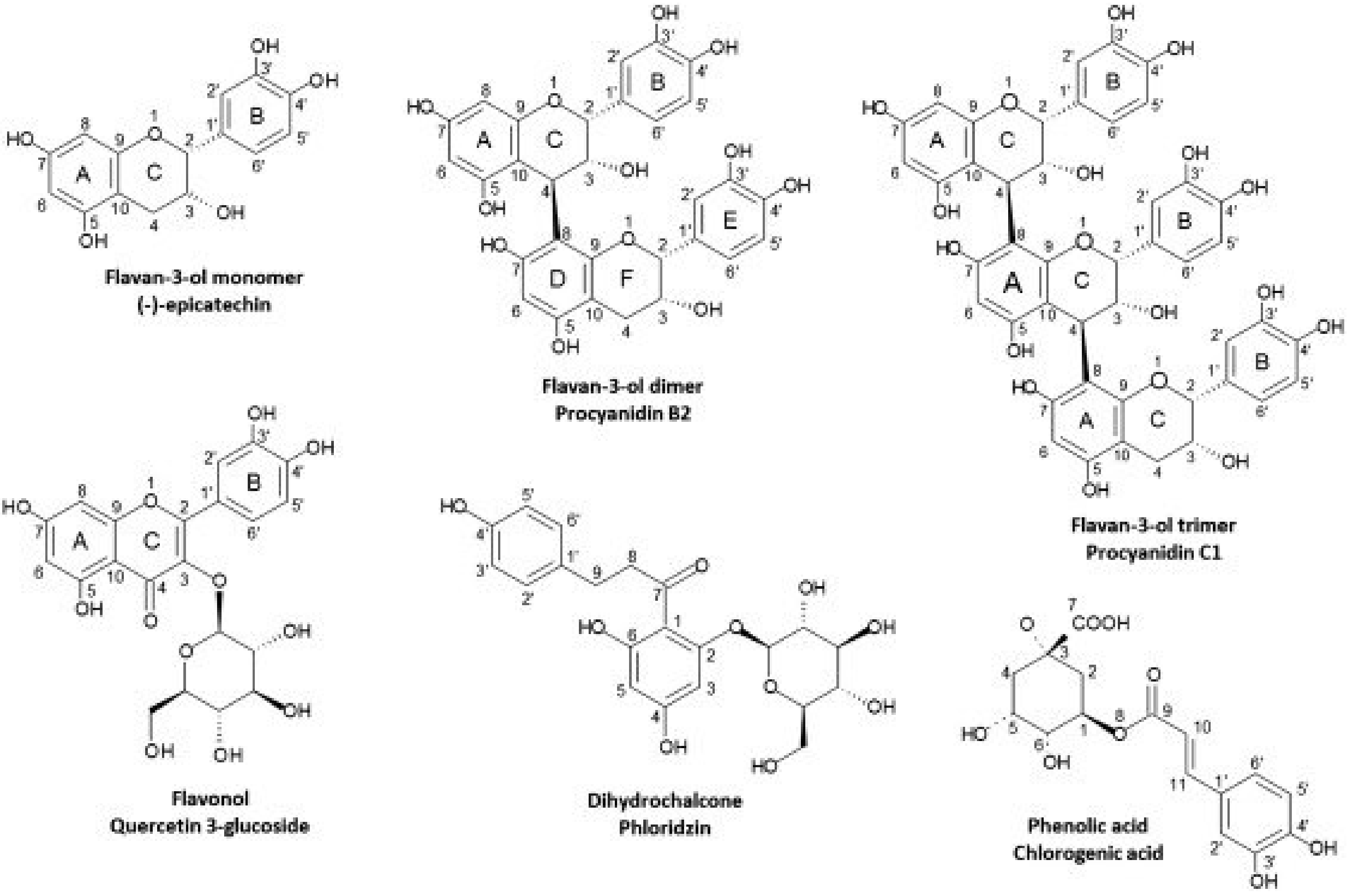


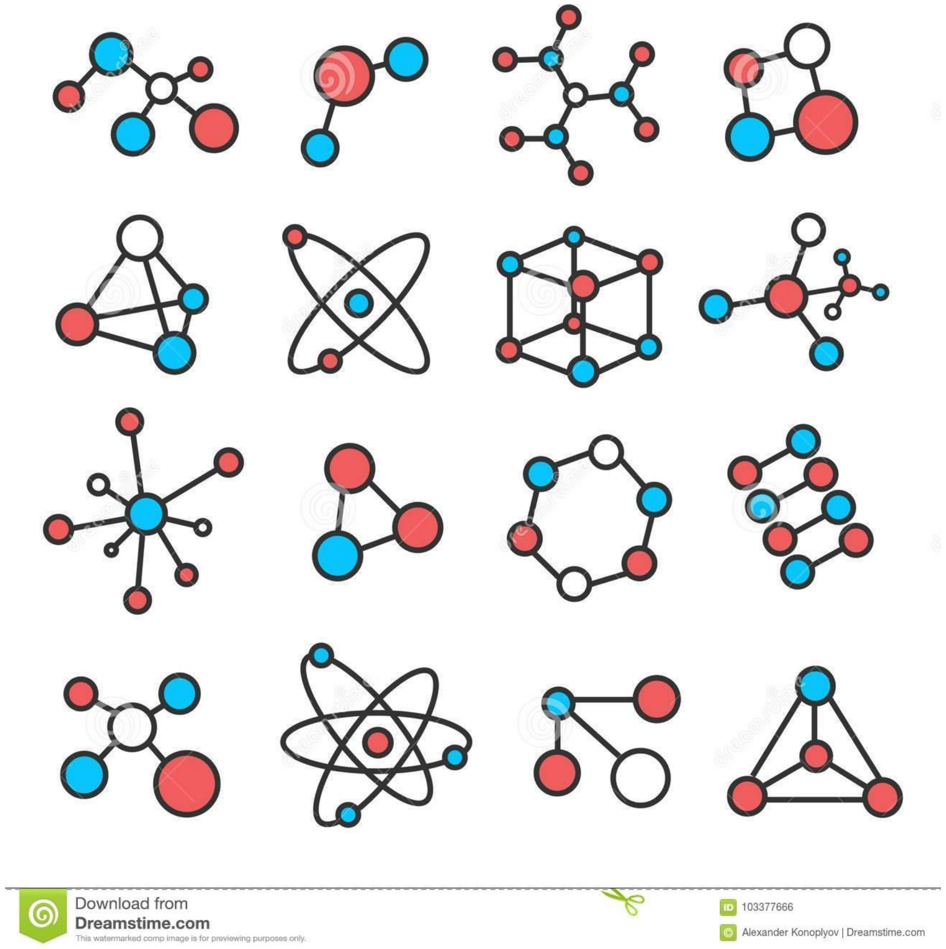
Gene structure

- Tree graph

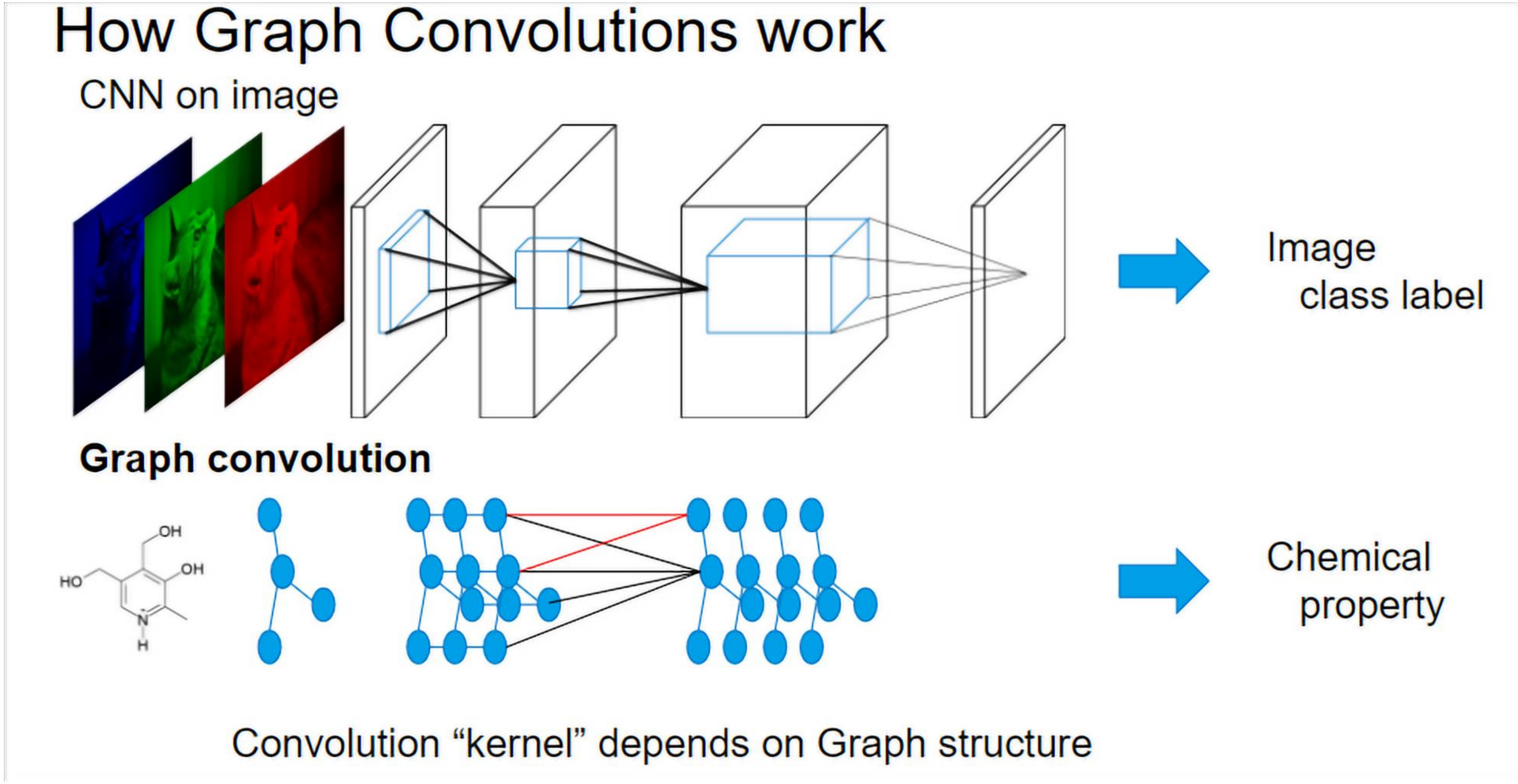


Molecular structure

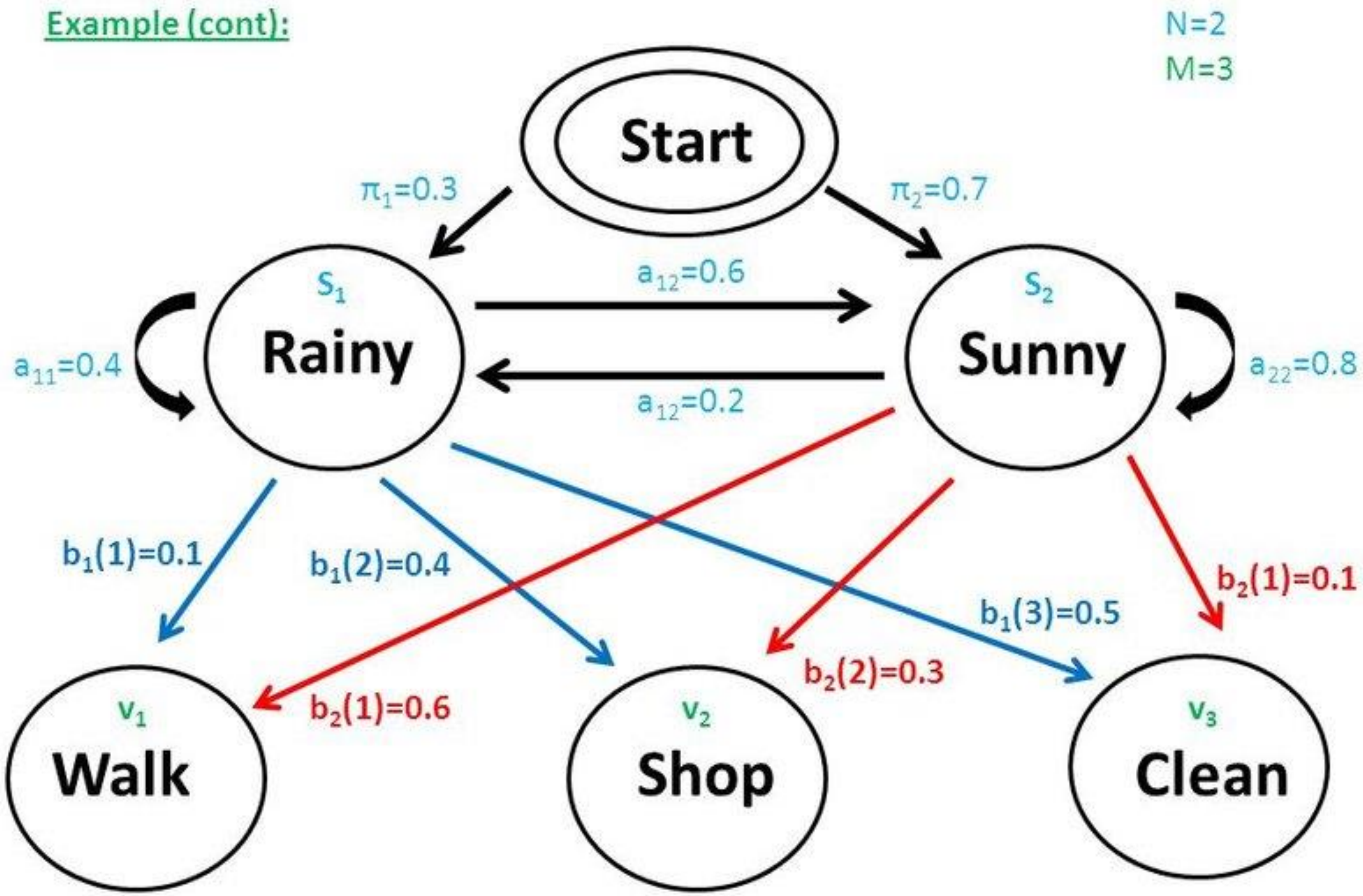




Graph neural network (GNN)



Hidden Markov Model



Basics

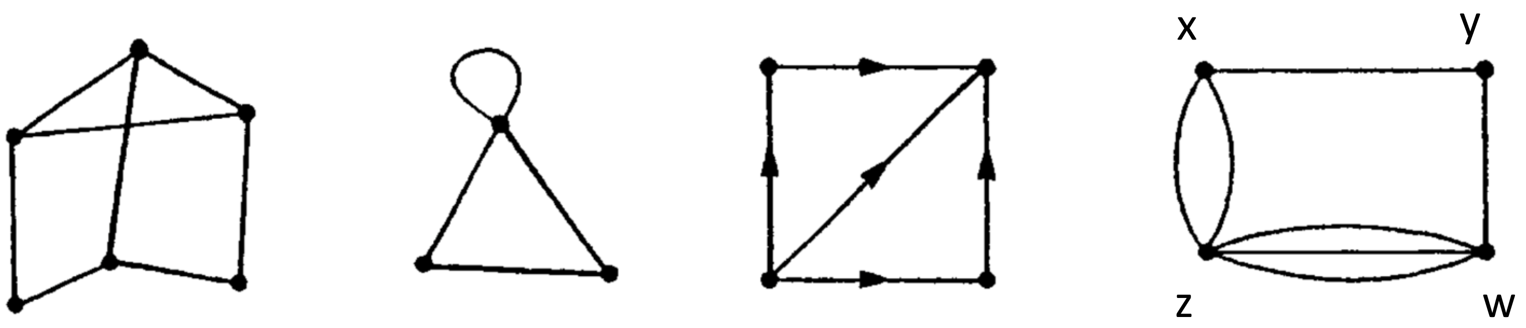
Graphs

- **Definition** A graph G is a pair (V, E)
  - V: set of vertices
  - E: set of edges
  - $e \in E$ corresponds to a pair of endpoints $x, y \in V$

We mainly focus on
Simple graph:
No loops, no multi-edges

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

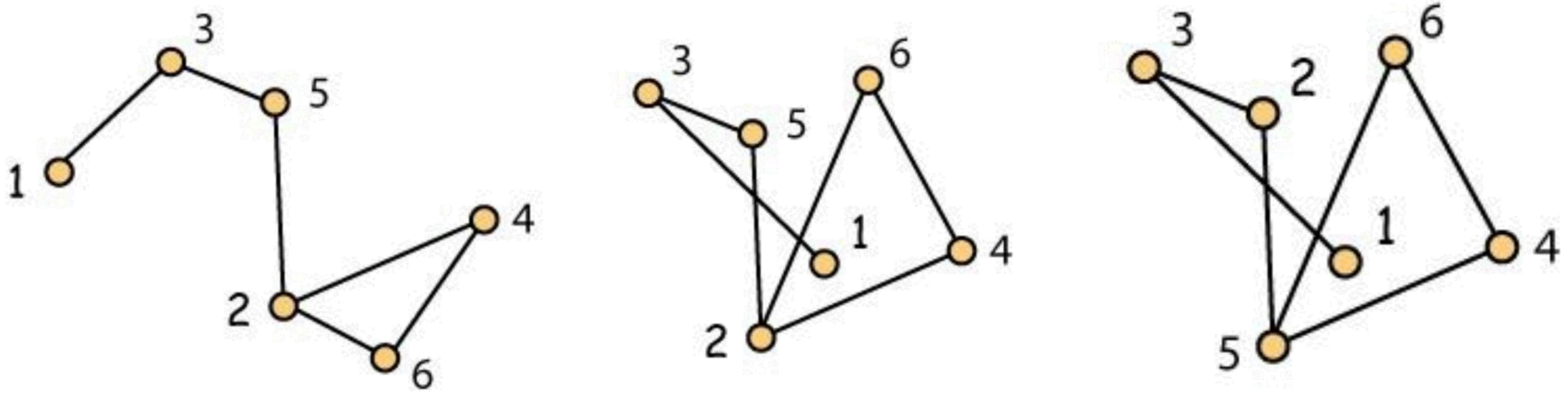


(i) graph (ii) graph with loop (iii) digraph (iv) multiple edges

Figure 1.2

Graphs: All about adjacency

- Same graph or not



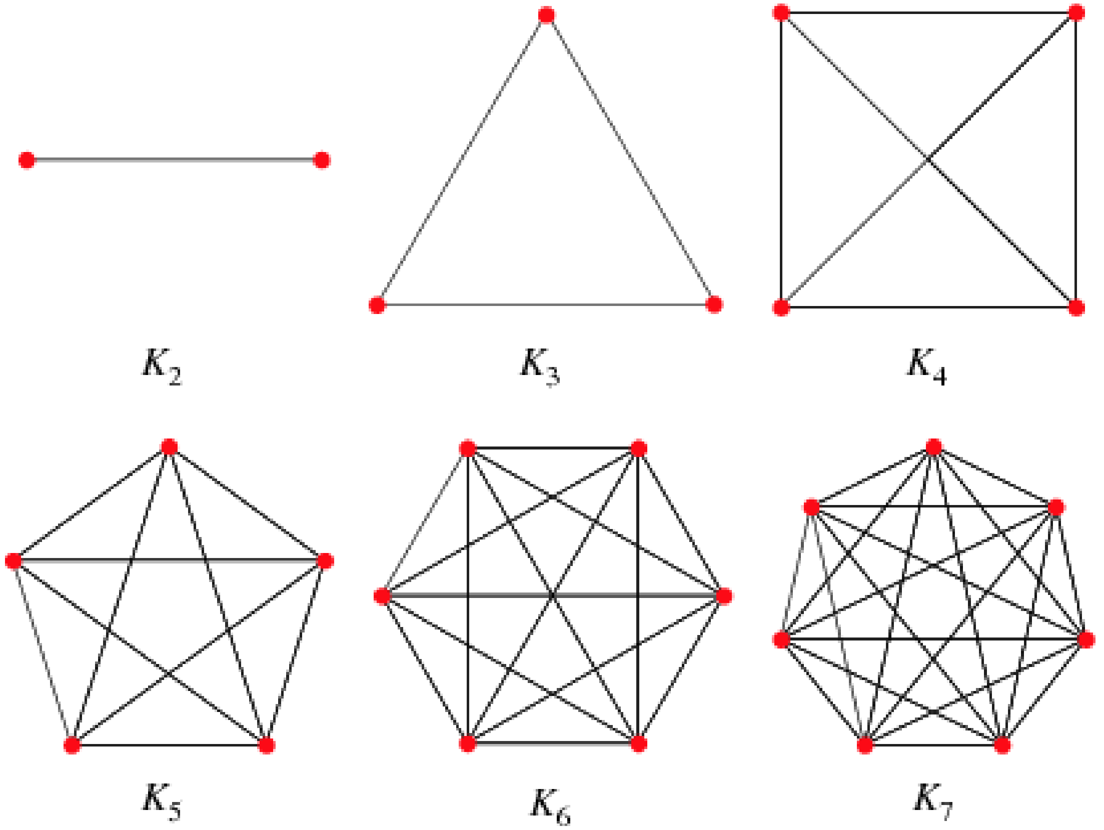
(a) (b) (c)

- Two graphs $G_1 = (V_1, E_1), G_1 = (V_2, E_2)$ are isomorphic if there is a bijection $f: V_1 \to V_2$ s.t.

$$e = \{a, b\} \in E_1 \Leftrightarrow f(e) := \{f(a), f(b)\} \in E_2$$

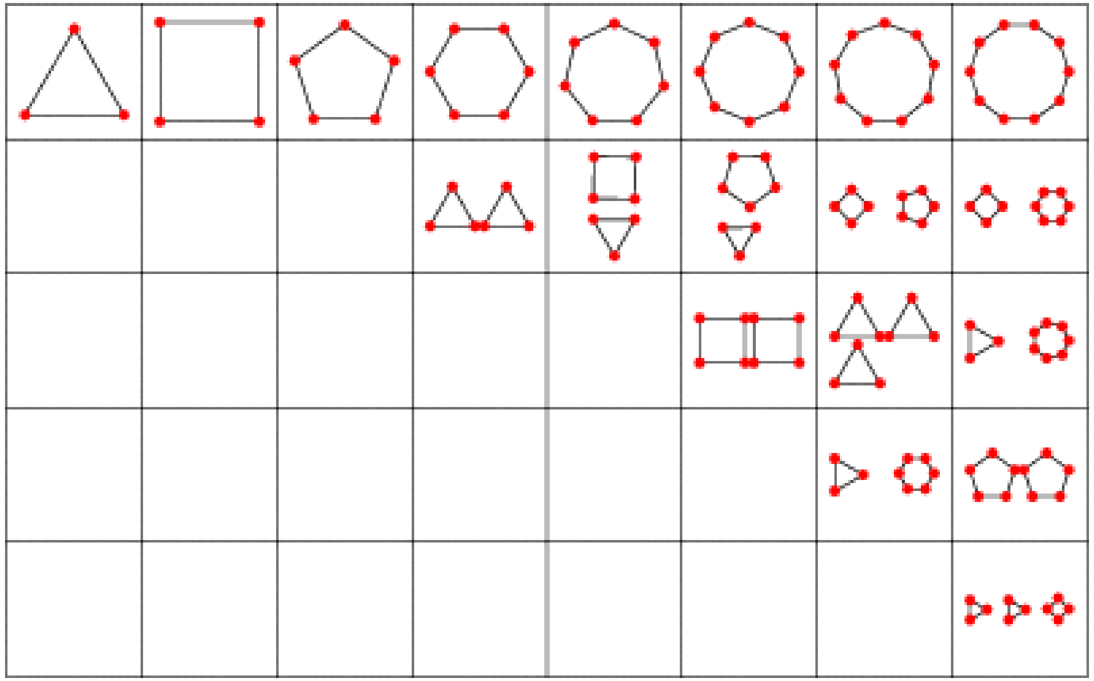
Example: Complete graphs

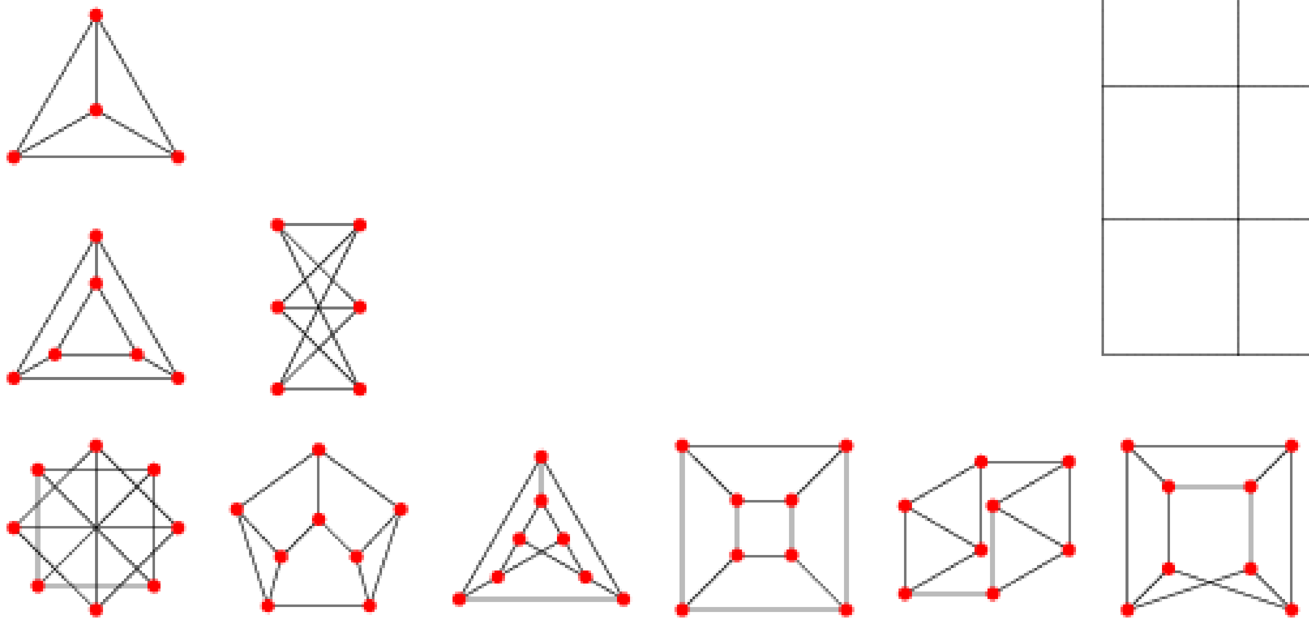
- There is an edge between every pair of vertices



Example: Regular graphs

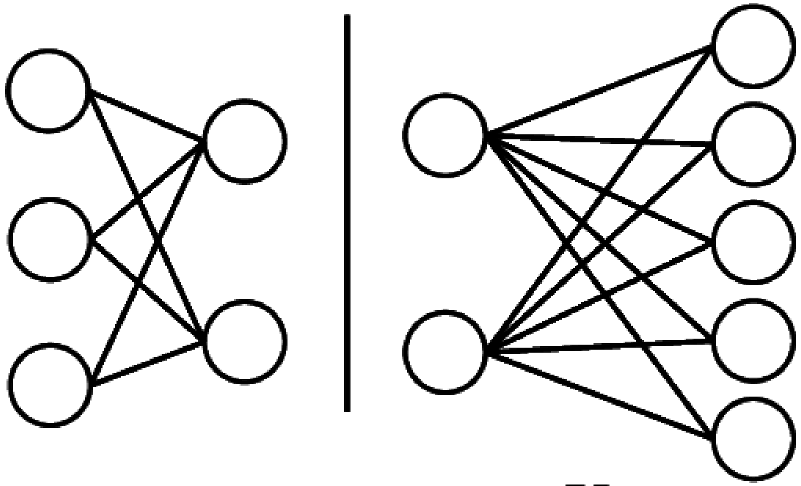
- Every vertex has the same degree





Example: Bipartite graphs

- The vertex set can be partitioned into two sets X and Y such that every edge in G has one end vertex in X and the other in Y
- Complete bipartite graphs



$\mathbf{K_{3,2}}$ $\mathbf{K_{2,5}}$

Example (1A, L): Peterson graph

- Show that the following two graphs are same/isomorphic

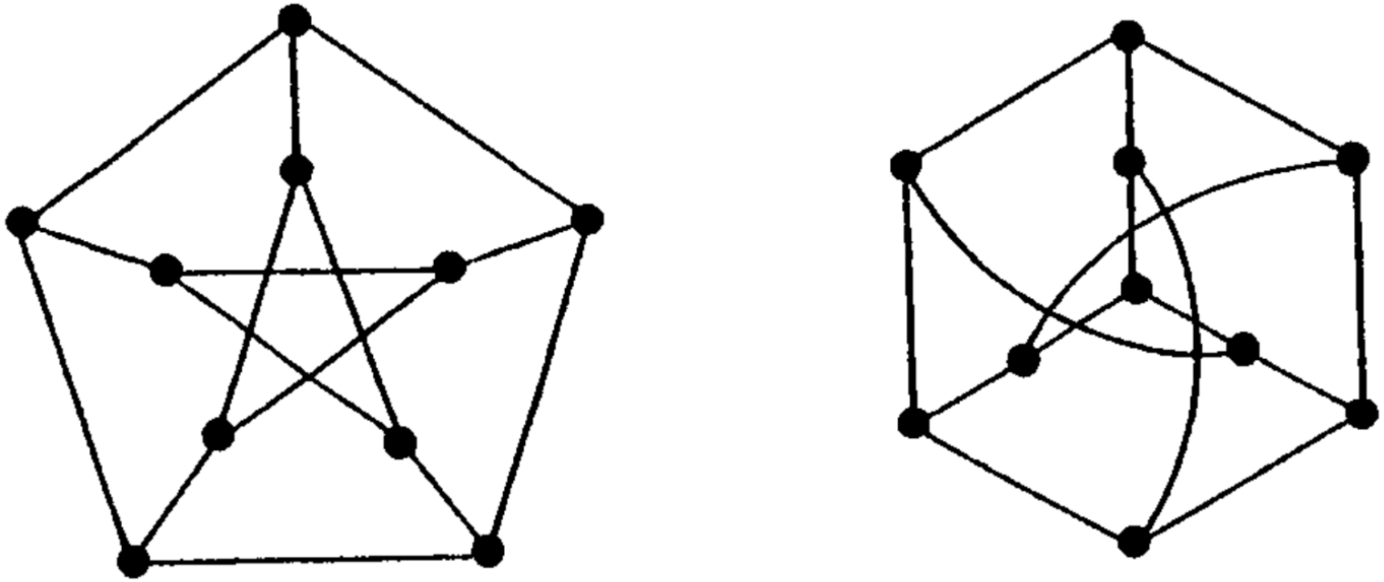
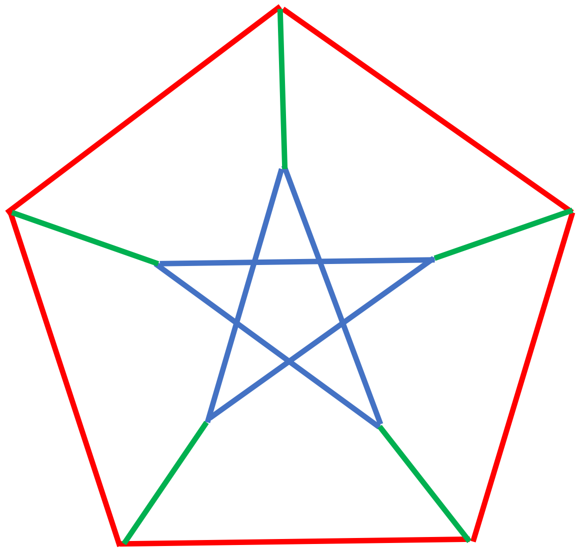
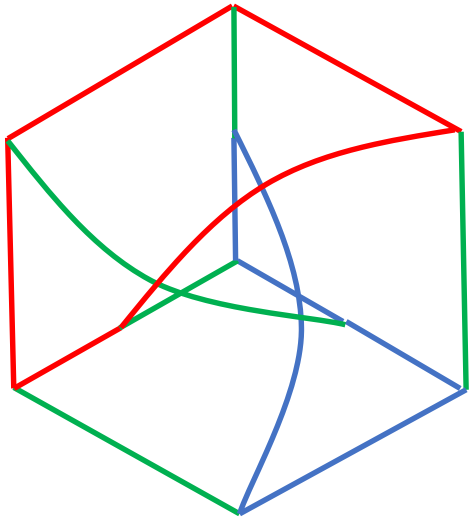


Figure 1.4

Example: Peterson graph (cont.)

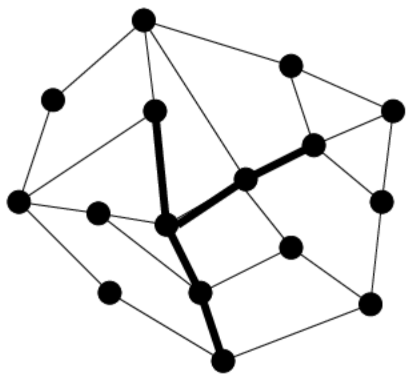
- Show that the following two graphs are same/isomorphic



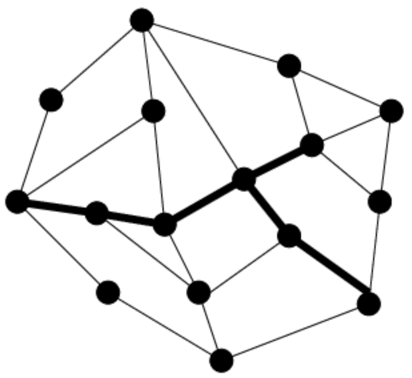


Subgraphs

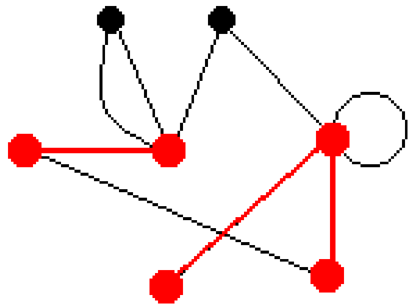
- A subgraph of a graph G is a graph H such that
$$V(H) \subseteq V(G), E(H) \subseteq E(G)$$
and the ends of an edge $e \in E(H)$ are the same as its ends in G
  - H is a spanning subgraph when $V(H) = V(G)$
  - The subgraph of G induced by a subset $S \subseteq V(G)$ is the subgraph whose vertex set is S and whose edges are all the edges of G with both ends in S



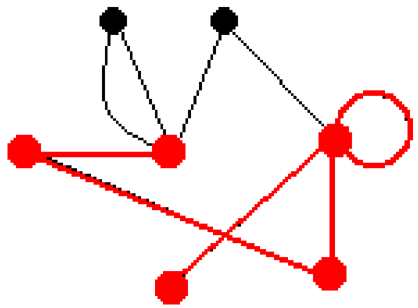
(a)



(b)



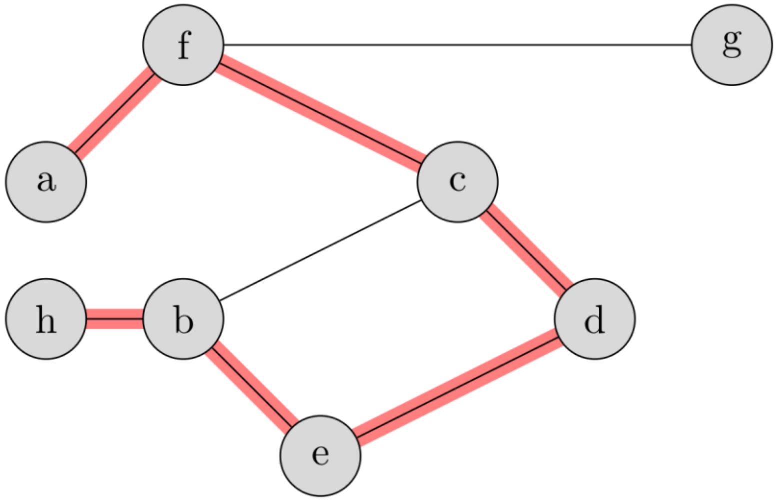
Subgraph (in red)



Induced Subgraph

Paths (路径)

- A path is a nonempty graph $P = (V, E)$ of the form
$$V = \{x_0, x_1, \dots, x_k\} \quad E = \{x_0x_1, x_1x_2, \dots, x_{k-1}x_k\}$$
where the x_i are all distinct
- P^k: path of length k (the number of edges)

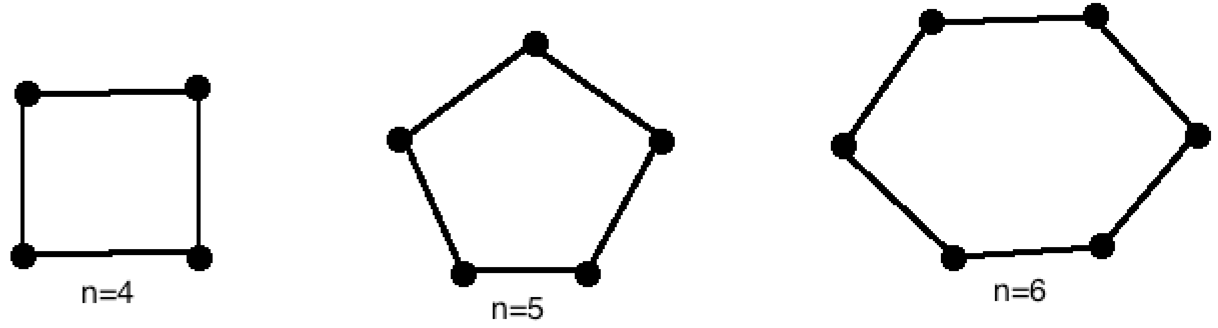


Walk (游走)

- A walk is a non-empty alternating sequence $v_0 e_1 v_1 e_2 \dots e_k v_k$
  - The vertices not necessarily distinct
  - The length = the number of edges
- Proposition (1.2.5, W) Every u-v walk contains a u-v path

Cycles (环)

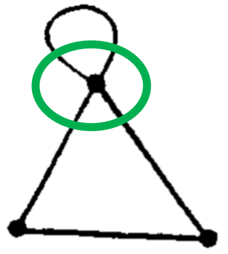
- If $P = x_0 x_1 \ldots x_{k-1}$ is a path and $k \geq 3$, then the graph $C := P + x_{k-1}x_0$ is called a cycle
- C^k: cycle of length k (the number of edges/vertices)



- Proposition (1.2.15, W) Every closed odd walk contains an odd cycle

Neighbors and degree

- Two vertices $a \neq b$ are called adjacent if they are joined by an edge
  - $N(x)$: set of all vertices adjacent to x
    - neighbors of x
  - A vertex is isolated vertex if it has no neighbors
- The number of edges incident with a vertex x is called the degree of x
  - A loop contributes 2 to the degree



graph with loop

- A graph is finite when both $E(G)$ and $V(G)$ are finite sets

Handshaking Theorem (Euler 1736)

- Theorem A finite graph G has an even number of vertices with odd degree.

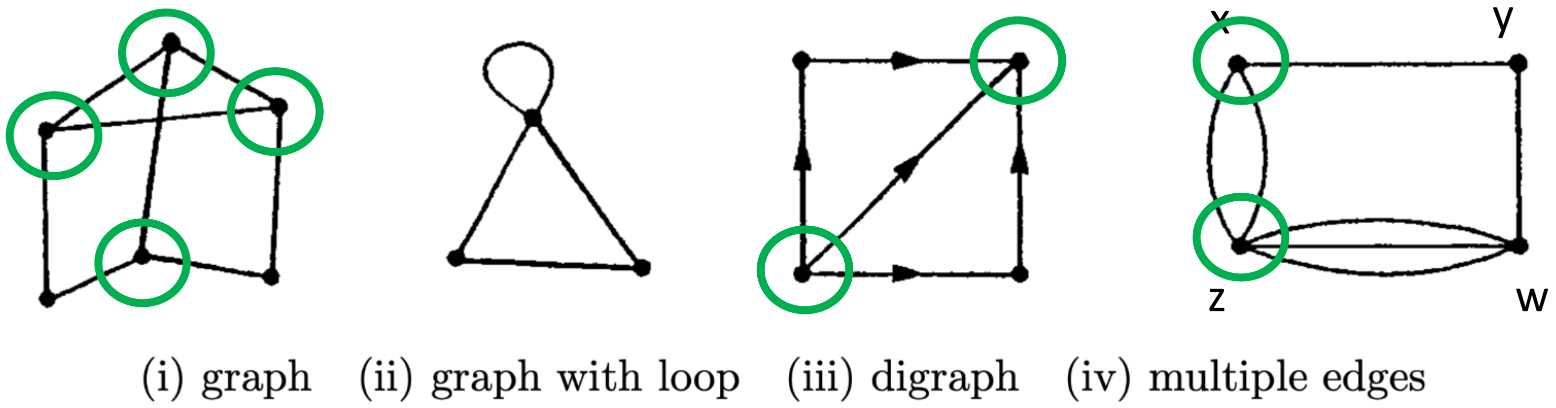


Figure 1.2

Proof

- Theorem A finite graph G has an even number of vertices with odd degree.
- Proof The degree of x is the number of times it appears in the right column. Thus

$$\sum_{x \in V(G)} \deg(x) = 2|E(G)|$$

edge	ends
a	x, z
b	y, w
c	x, z
d	z, w
e	z, w
f	x, y
g	z, w

Figure 1.1

Degree

- Minimal degree of G: $\delta(G) = \min\{d(v): v \in V\}$
- Maximal degree of G: $\Delta(G) = \min\{d(v): v \in V\}$
- Average degree of G: $d(G) = \frac{1}{|V|}\sum_{v\in V} d(v) = \frac{2|E|}{|V|}$

- All measures the `density' of a graph

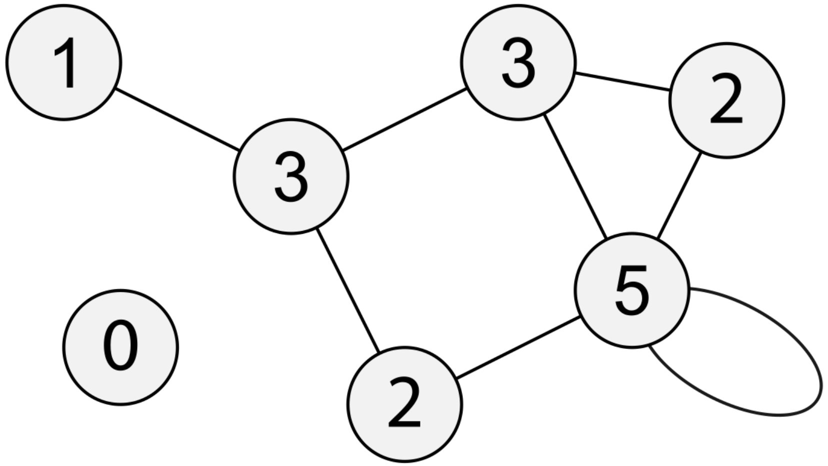
- $d(G) \geq \delta(G)$

Degree (global to local)

- Proposition (1.2.2, D) Every graph G with at least one edge has a subgraph H with

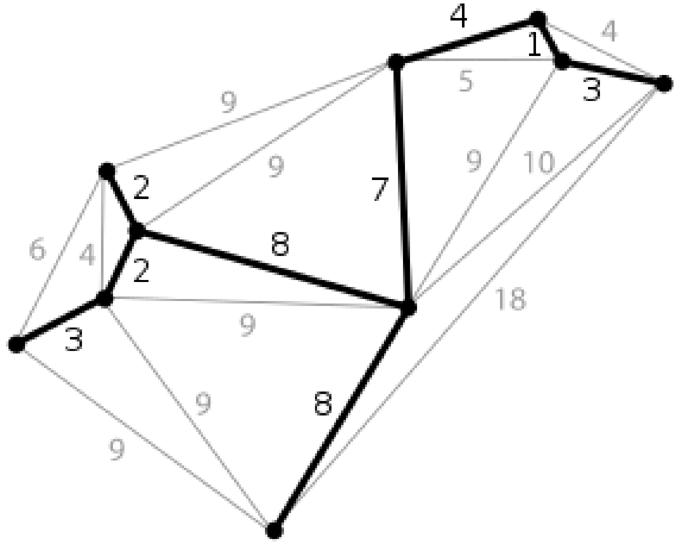
$$\delta(H) > \frac{1}{2}d(H) \geq \frac{1}{2}d(G)$$

- Example: $|G| = 7, d(G) = 16/7$



Minimal degree guarantees long paths and cycles

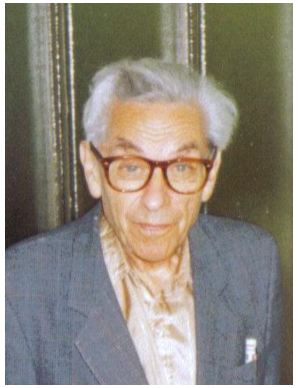
- Proposition (1.3.1, D) Every graph G contains a path of length $\delta(G)$ and a cycle of length at least $\delta(G)+1$, provided $\delta(G) \geq 2$.

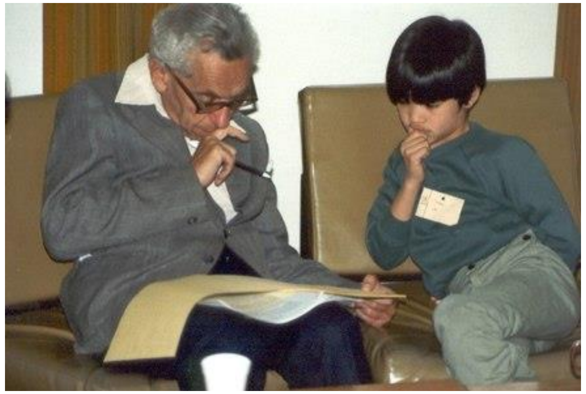


Distance and diameter

- The distance $d_G(x, y)$ in G of two vertices x, y is the length of a shortest $x \sim y$ path
  - if no such path exists, we set $d(x, y) \coloneqq \infty$
- The greatest distance between any two vertices in G is the diameter of G

Example -- Erdős number





- A well-known graph
  - vertices: mathematicians of the world
  - Two vertices are adjacent if and only if they have published a joint paper
  - The distance in this graph from some mathematician to the vertex Paul Erdős is known as his or her Erdős number