Lecture 6: Coloring

Shuai Li

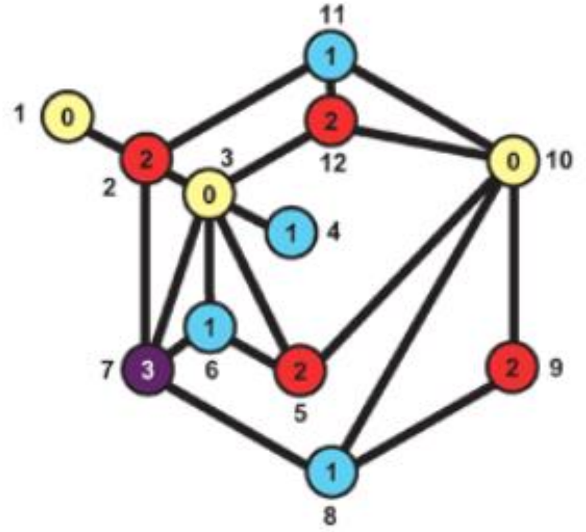
John Hopcroft Center, Shanghai Jiao Tong University

https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Motivation: Scheduling and coloring

- University examination timetabling
  - Two courses linked by an edge if they have the same students
- Meeting scheduling
  - Two meetings are linked if they have same member



Definitions

- Given a graph G and a positive integer k, a k-coloring is a function $K: V(G) \longrightarrow \{1, \dots, k\}$ from the vertex set into the set of positive integers less than or equal to k. If we think of the latter set as a set of k "colors," then K is an assignment of one color to each vertex.
- We say that K is a proper k-coloring of G if for every pair u, v of adjacent vertices, $K(u) \neq K(v)$ — that is, if adjacent vertices are colored differently. If such a coloring exists for a graph G, we say that G is k-colorable

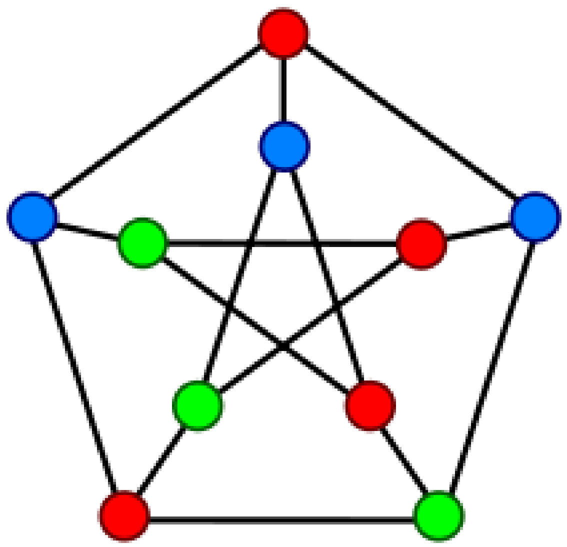
Chromatic number

- Given a graph G, the chromatic number of G, denoted by $\chi(G)$, is the smallest integer k such that G is k-colorable
- Examples

$$\chi(C_n) = \begin{cases} 2 & \text{if } n \text{ is even,} \\ 3 & \text{if } n \text{ is odd,} \end{cases}$$

$$\chi(P_n) = \begin{cases} 2 & \text{if } n \geq 2, \\ 1 & \text{if } n = 1, \end{cases}$$

$$\chi(K_n) = n,$$
$$\chi(E_n) = 1,$$
$$\chi(K_{m,n}) = 2.$$



- (Ex5, S1.6.1, H) A graph G of order at least two is bipartite $\Leftrightarrow$ it is 2-colorable

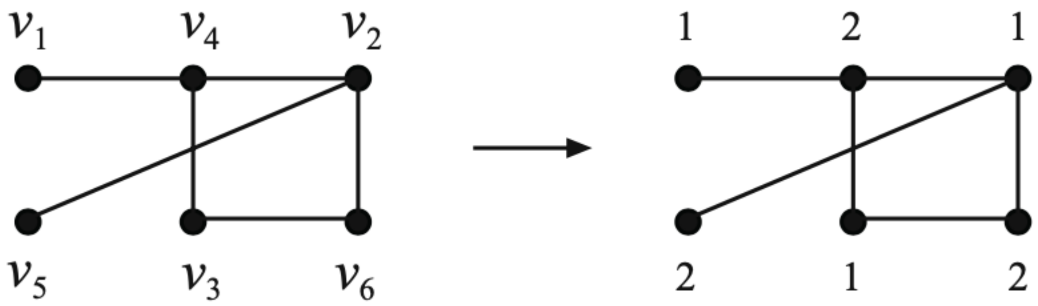
Bounds on Chromatic number

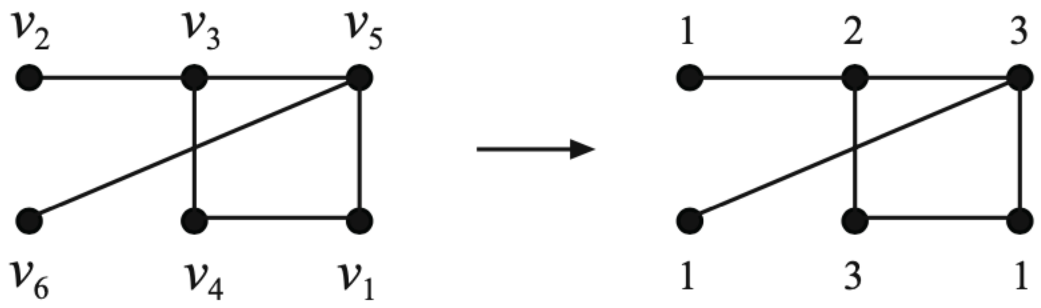
- Theorem (1.41, H) For any graph G of order n, $\chi(G) \leq n$
- It is tight since $\chi(K_n) = n$
- $\chi(G) = n \Leftrightarrow G = K_n$

Greedy algorithm

- First label the vertices in some order—call them $v_1, v_2, \dots, v_n$
- Next, order the available colors $(1,2,\dots,n)$ in some way
    - Start coloring by assigning color 1 to vertex v_1
    - If v_1 and v_2 are adjacent, assign color 2 to vertex v_2; otherwise, use color 1
    - To color vertex v_i, use the first available color that has not been used for any of v_i's previously colored neighbors

Examples: Different orders result in different number of colors



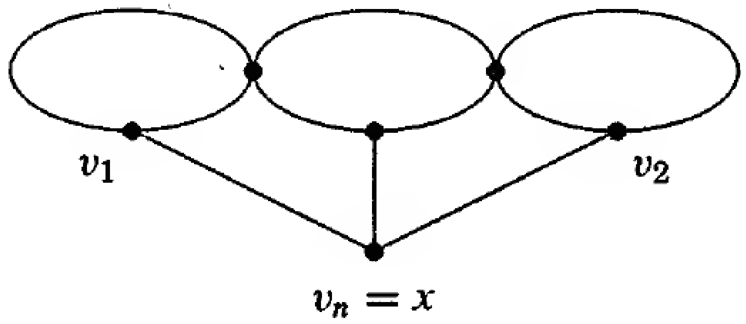


Bound of the greedy algorithm

- Theorem (1.42, H) For any graph G, $\chi(G) \leq \Delta(G) + 1$
- The equality is obtained for complete graphs and cycles with an odd number of vertices

Brooks's theorem

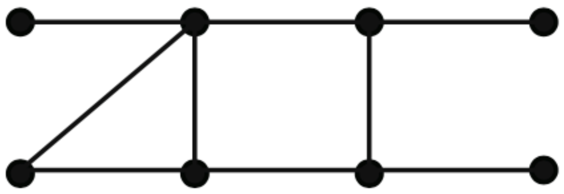
- Theorem (1.43, H; 5.1.22, W; 5.2.4, D; Brooks 1941)
  If G is a connected graph that is neither an odd cycle or a complete graph, then $\chi(G) \leq \Delta(G)$



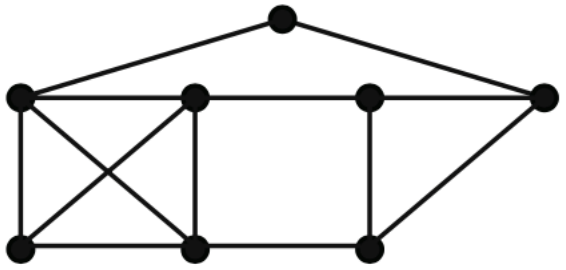
Chromatic number and clique number

- The clique number $\omega(G)$ of a graph is defined as the order of the largest complete graph that is a subgraph of G
- Example: $\omega(G_1) = 3, \omega(G_2) = 4$

G_1



G_2



- Theorem (1.44, H) For any graph G, $\chi(G) \geq \omega(G)$

Chromatic number and independence number

- Theorem (1.45, H; Ex6, S1.6.2, H) For any graph G of order n,
$$\frac{n}{\alpha(G)} \leq \chi(G) \leq n + 1 - \alpha(G)$$

The Four Color Problem

- Q: Is it true that the countries on any given map can be colored with four or fewer colors in such a way that adjacent countries are colored differently?
- Theorem (Four Color Theorem) Every planar graph is 4-colorable
- Theorem (Five Color Theorem) (1.47, H) Every planar graph is 5-colorable

Theorem 1.35. *If G is a planar graph, then G contains a vertex of degree at most five. That is, $\delta(G) \leq 5$.*

Chromatic Polynomials

Definition and examples

- It is brought up by George David Birkhoff in 1912 in an attempt to prove the four color theorem
- Define $c_G(k)$ to be the number of different colorings of a graph G using at most k colors
- Examples:
  - How many different colorings of K_4 using 4 colors?
    - $4 \times 3 \times 2 \times 1$
    - $c_{K_4}(4) = 24$
  - How many different colorings of K_4 using 6 colors?
    - $6 \times 5 \times 4 \times 3$
    - $c_{K_4}(6) = 360$
  - How many different colorings of K_4 using 2 colors?
    - 0
    - $c_{K_4}(2) = 0$

Examples

- If $k \geq n$

$$c_{K_n}(k) = k(k-1)\cdots(k-n+1)$$

- If $k < n$

$$c_{K_n}(k) = 0$$

- G is k-colorable $\Leftrightarrow$ $\chi(G) \leq k$ $\Leftrightarrow c_G(k) > 0$
- $\chi(G) = \min\{k \geq 1 : c_G(k) > 0\}$

Chromatic recurrence

- $G - e$ and G/e

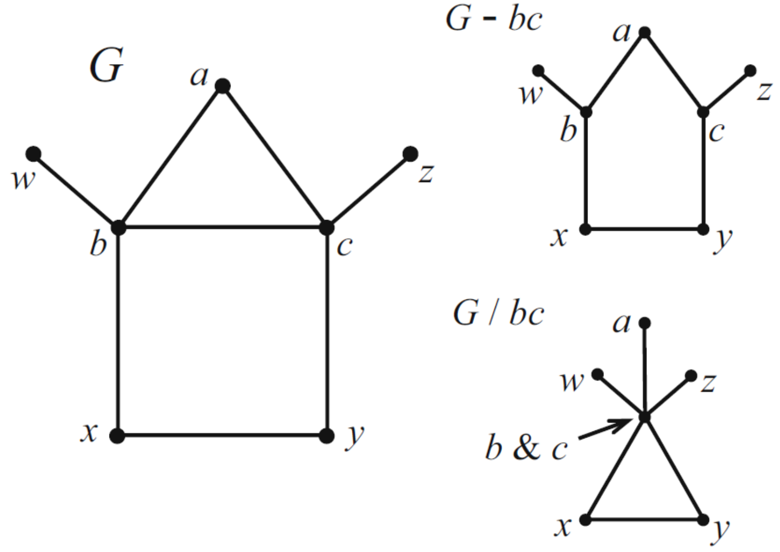


FIGURE 1.98. Examples of the operations.

- Theorem (1.48, H; 5.3.6, W) Let G be a graph and e be any edge of G. Then

$$c_G(k) = c_{G-e}(k) - c_{G/e}(k)$$

Use chromatic recurrence to compute $c_G(k)$

- Example: Compute $c_{P_3}(k) = k^4 - 3k^3 + 3k^2 - k$
- Check: $c_{P_3}(1) = 0, c_{P_3}(2) = 2$

FIGURE 1.102. Two 2-colorings of P_3

More examples

- Path P_{n-1} has $n - 1$ edges (n vertices)

$$c_{P_{n-1}}(k) = k(k-1)^{n-1}$$

- Any tree T on n vertices

$$c_T(k) = k(k-1)^{n-1}$$

- Cycle C_n

$$c_{C_n}(k) = (k-1)^n + (-1)^n(k-1)$$

  - When n is odd, $c_{C_n}(2) = 0, c_{C_n}(3) > 0$
  - When n is even, $c_{C_n}(2) > 0$

Properties of chromatic polynomials

- Theorem (1.49, H; Ex 3, S1.6.4, H) Let G be a graph of order n
  - $c_G(k)$ is a polynomial in k of degree n
  - The leading coefficient of $c_G(k)$ is 1
  - The constant term of $c_G(k)$ is 0
    - If G has i components, then the coefficients of $k^0, \ldots, k^{i-1}$ are 0
    - G is connected $\Leftrightarrow$ the coefficient of k is nonzero
  - The coefficients of $c_G(k)$ alternate in sign
  - The coefficient of the k^{n-1} term is $-|E(G)|$
    - A graph G is a tree $\Leftrightarrow c_G(k) = k(k-1)^{n-1}$

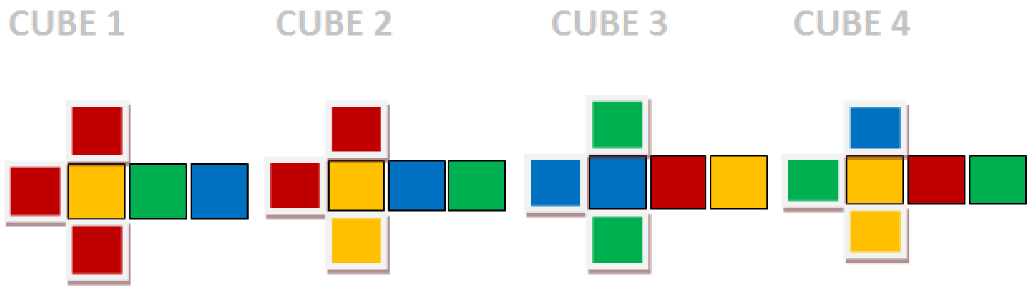
      $\Leftrightarrow$ (Theorem 1.10, 1.12, H) T is connected with $n-1$ edges

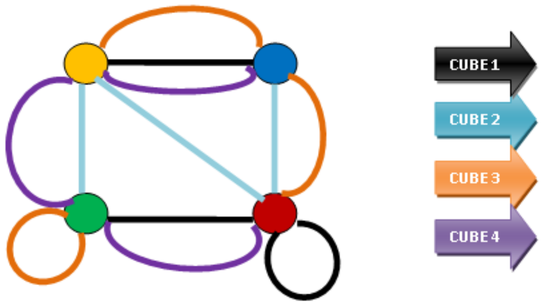
    - A graph G is complete $\Leftrightarrow c_G(k) = k(k-1)\cdots(k-n+1)$

Proof Using Coloring

Example -- Instant Insanity 四色方柱问题 (1.2, L)

- Problem make a stack of these cubes so that all four colors appear on each of the four sides of the stack
- An edge indicates that the two adjacent colors occur on opposite faces of the cube
- Problem necessary to find two subgraphs s.t.
  - are regular of degree 2
  - four edges from each cube
  - no edge in common





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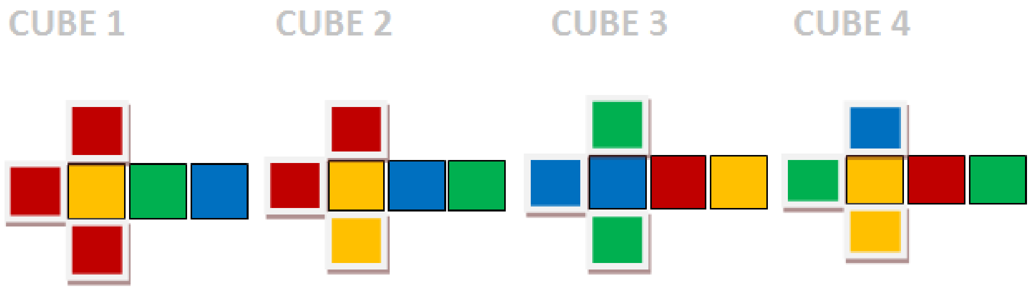

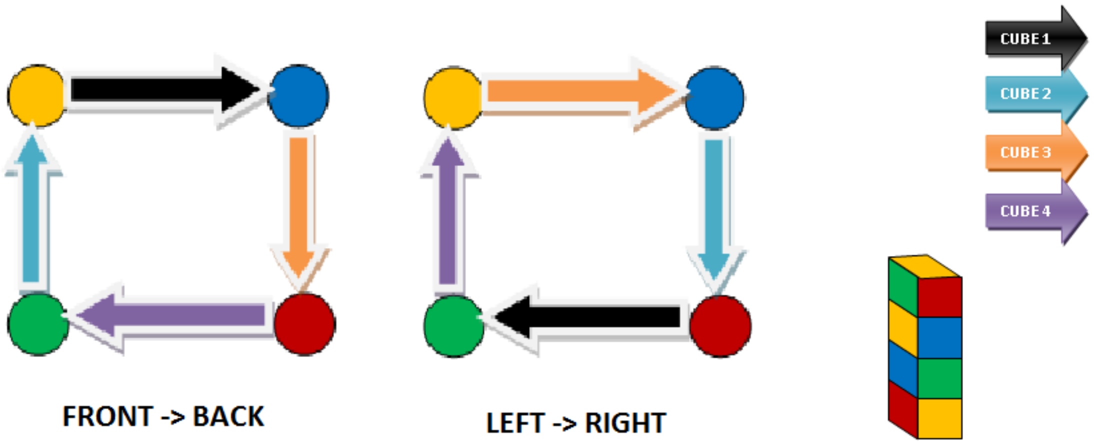

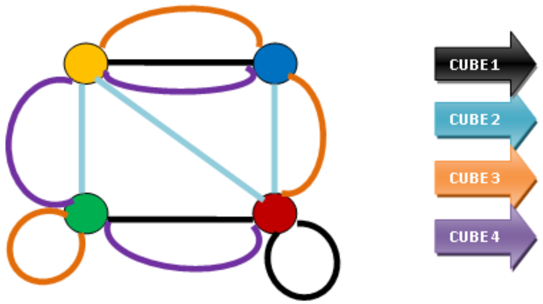

An example about sets (1E, L)

- Let $A_1, \dots, A_n$ be n distinct subsets of the n-set $N := \{1, \dots, n\}$. Show that there is an element $x \in N$ such that the sets $A_i \setminus \{x\}, 1 \le i \le n$, are all distinct
- Proof Consider a graph with vertices $A_1, \dots, A_n$.
  - An edge of `color’ x between A_i and A_j iff $A_i \Delta A_j = \{x\}$
  - Then the problem is equivalent to find y s.t. no color y
  - Notice that a cycle in this graph must have even length and each color appears even times
  - Then we can remove an edge if there is an edge with same color
  - Thus the number of colors remain the same and no cycle exists
  - By tree property, the number of edges is at most $n - 1$