Lecture 7: Planarity

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https://shuaili8.github.io

https://shuaili8.github.io/Teaching/CS445/index.html

Motivation

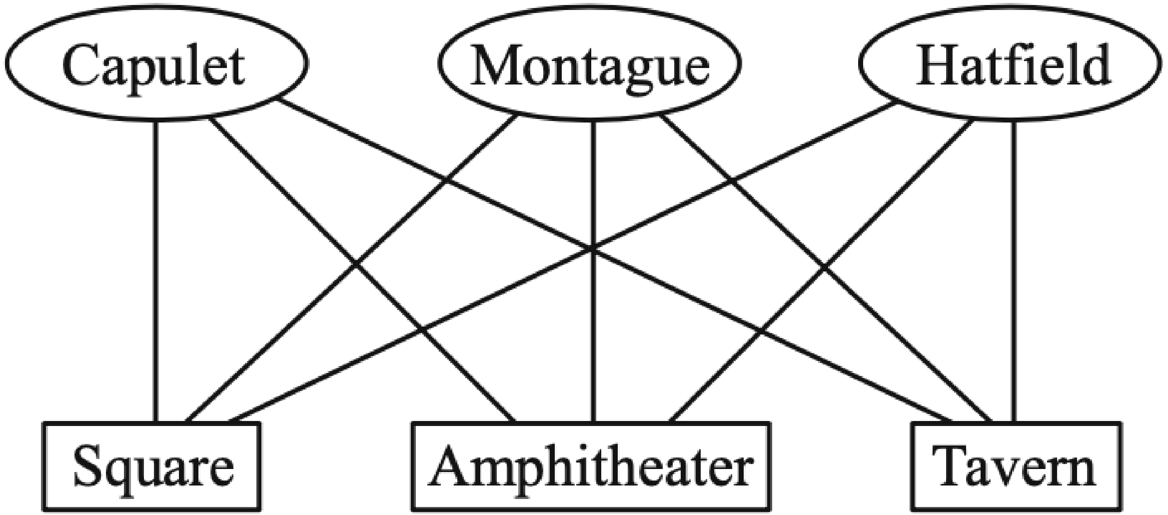


FIGURE 1.72. Original routes.

Definition and examples

- A graph G is said to be planar if it can be drawn in the plane in such a way that pairs of edges intersect only at vertices
- If G has no such representation, G is called nonplanar
- A drawing of a planar graph G in the plane in which edges intersect only at vertices is called a planar representation (or a planar embedding) of G

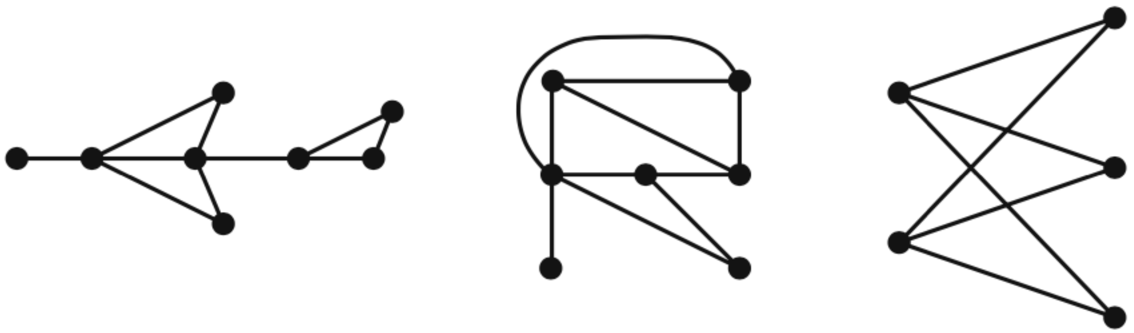
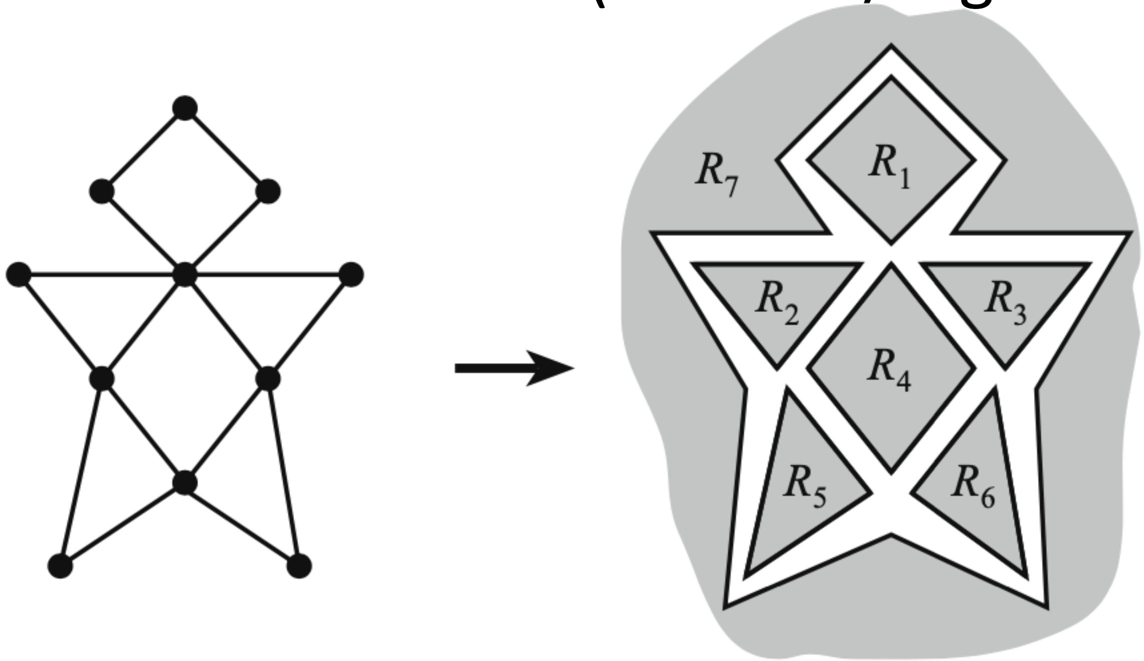


FIGURE 1.73. Examples of planar graphs.

Region

- Given a planar representation of a graph G, a region is a maximal section of the plane in which any two points can be joined by a curve that does not intersect any part of G
- The region R_7 is called the exterior (or outer) region



An edge bounds a region

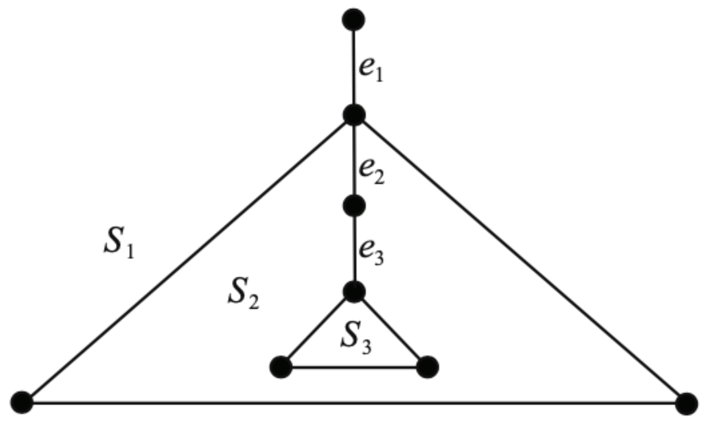
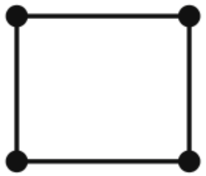


FIGURE 1.76. Edges e_1, e_2, and e_3 touch one region only.

- An edge can come into contact with either one or two regions
- Example:
  - Edge e_1 is only in contact with one region S_1
  - Edge e_2, e_3 are only in contact with S_2
  - Each of other edges is in contact with two regions
- An edge e bounds a region R if e comes into contact with R and with a region different from R
- The bounded degree $b(R)$ is the number of edges that bound the region
  - Example: $b(S_1) = b(S_3) = 3, b(S_2) = 6$

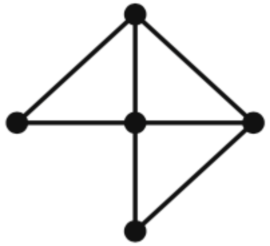
The relationship between numbers of vertices, edges and regions

- The number of vertices n
- The number of edges m
- The number of regions r

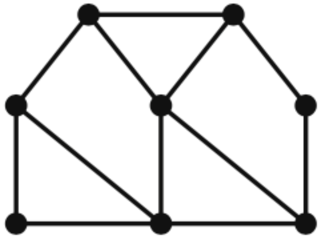


$n = 4$
$m = 4$
$r = 2$

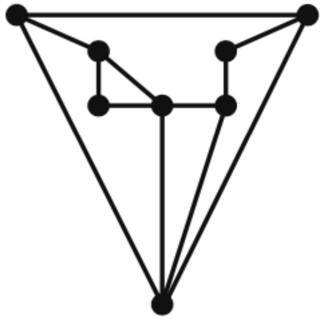
$n = 7$
$m = 9$
$r = 4$



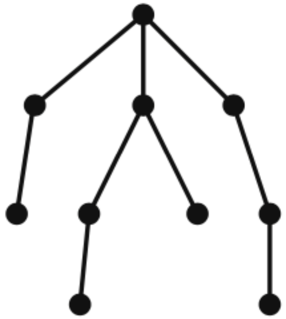
$n = 5$
$m = 7$
$r = 4$



$n = 8$
$m = 12$
$r = 6$



$n = 8$
$m = 12$
$r = 6$



$n = 10$
$m = 9$
$r = 1$

Euler’s formula

- Theorem (1.31, H; Euler 1748) If G is a connected planar graph with n vertices, m edges, and r regions, then
$$n - m + r = 2$$
  - Need Lemma: (Ex4, S1.5.1, H) Every tree is planar
- (Ex6, S1.5.2, H) Let G be a planar graph with k components. Then
$$n - m + r = k + 1$$

$K_{3,3}$ is nonplanar

- Theorem (1.32, H) $K_{3,3}$ is nonplanar

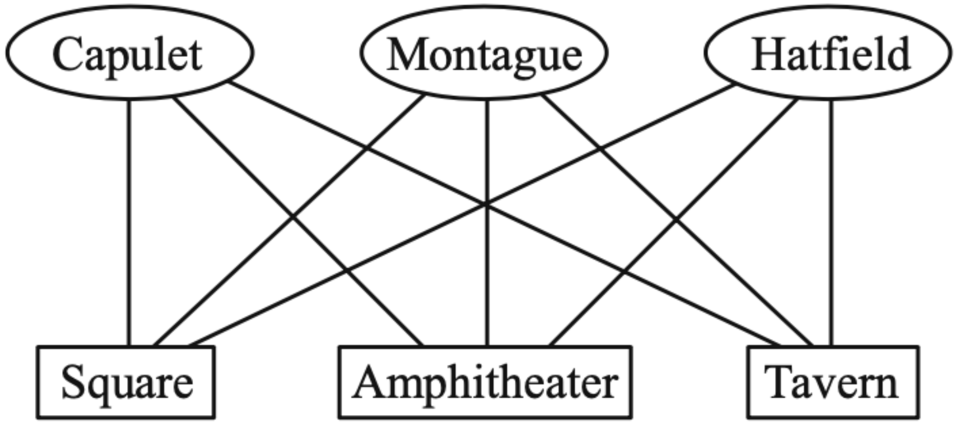


FIGURE 1.72. Original routes.

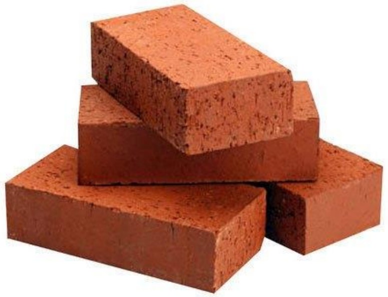
Upper bound for m

- Theorem (1.33, H) If G is a planar graph with $n \geq 3$ vertices and m edges, then $m \leq 3n - 6$. Furthermore, if equality holds, then every region is bounded by 3 edges.
- (Ex4, S1.5.2, H) Let G be a connected, planar, K_3-free graph of order $n \geq 3$. Then G has no more than $2n - 4$ edges
- Corollary (1.34, H) K_5 is nonplanar
- Theorem (1.35, H) If G is a planar graph , then $\delta(G) \leq 5$
- (Ex5, S1.5.2, H) If G is bipartite planar graph, then $\delta(G) < 4$

Polyhedra

(Convex) Polyhedra 多面体

- A polyhedron is a solid that is bounded by flat surfaces

Polyhedra are planar

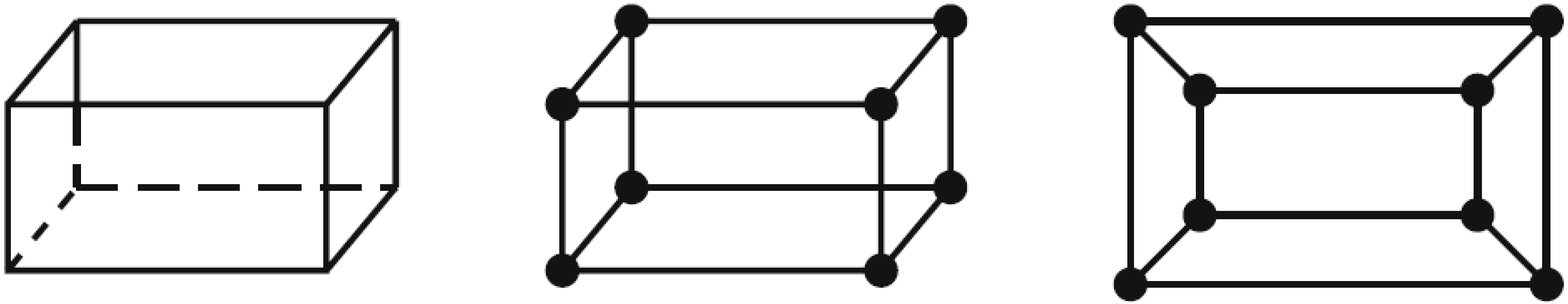


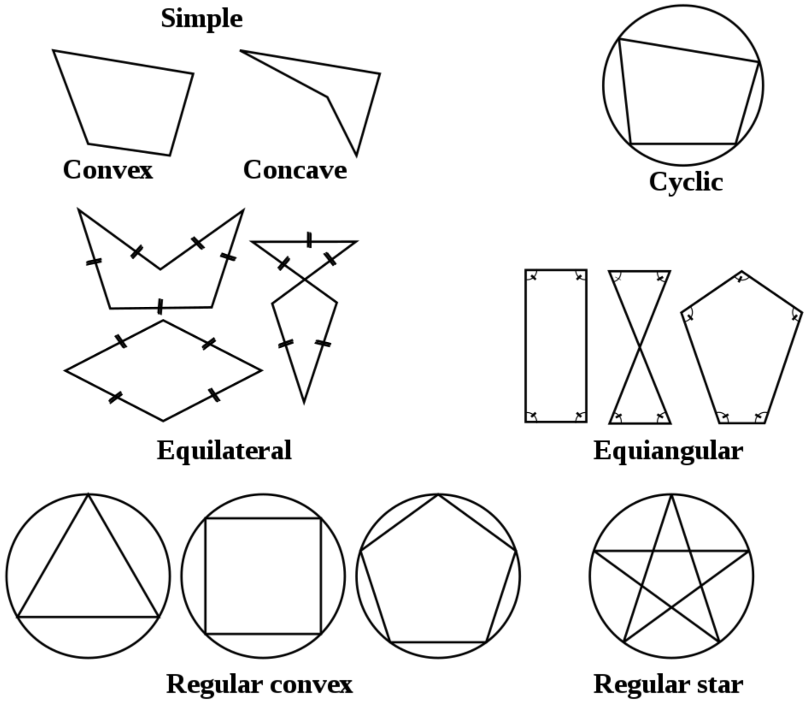
FIGURE 1.81. A polyhedron and its graph.

Properties

- Theorem (1.36, H) If a polyhedron has V vertices, E edges, and F faces, then
$$V - E + F = 2$$
- Given a polyhedron P, define
$$\rho(P) = \min\{b(R) \colon R \text{ is a region of } P\}$$
- Theorem (1.37, H) For all polyhedron P, $3 \leq \rho(P) \leq 5$

Regular polyhedron 正多面体

- A regular polygon is one that is equilateral and equiangular
  正多边形(cycle)，等边、等角
- A polyhedron is regular if its faces are mutually congruent, regular polygons and if the number of faces meeting at a vertex is the same for every vertex
  正多面体
  面是相互全等的、正多边形、点的度数相等



Regular polyhedron 正多面体

- Theorem (1.38, H) There are exactly five regular polyhedral
- 正四面体
- 立方体（正六面体）
- 正八面体
- 正十二面体
- 正二十面体

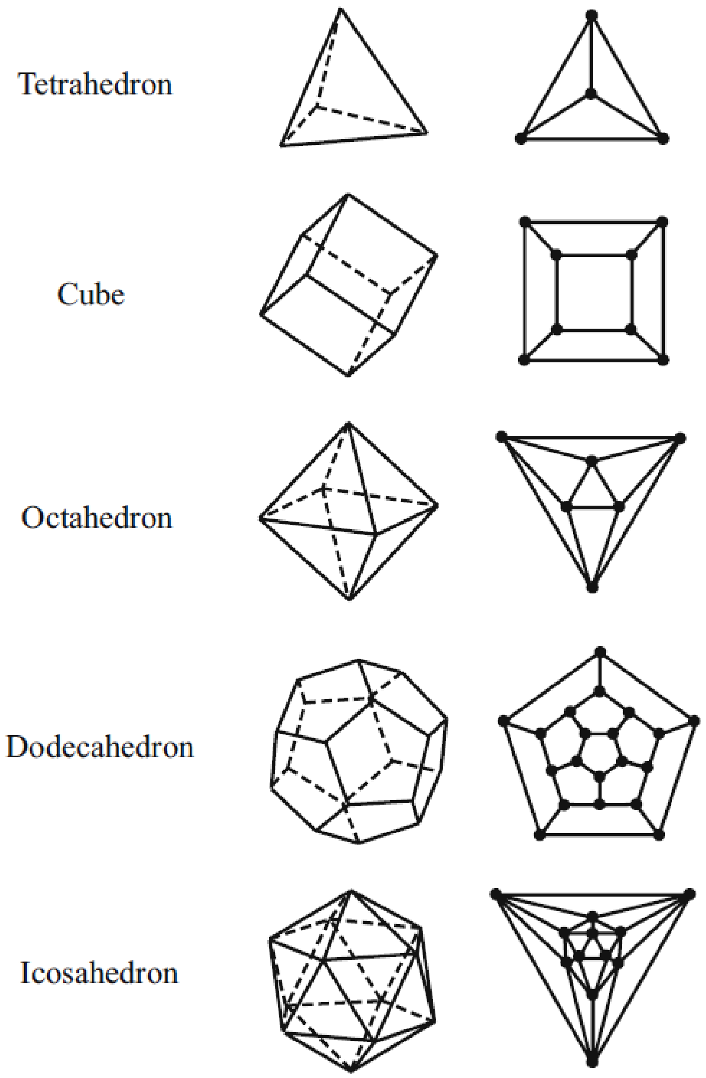
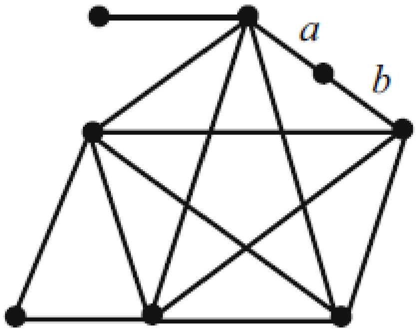


FIGURE 1.82. The five regular polyhedra and their graphical representations.

Kuratowski's Theorem

Subdivision 细分

- A subdivision of an edge e in G is a substitution of a path for e
- A graph H is a subdivision of G if H can be obtained from G by a finite sequence of subdivisions
- Example:
  - The graph on the right contains a subdivision of K_5
  - In the below, H is a subdivision of G



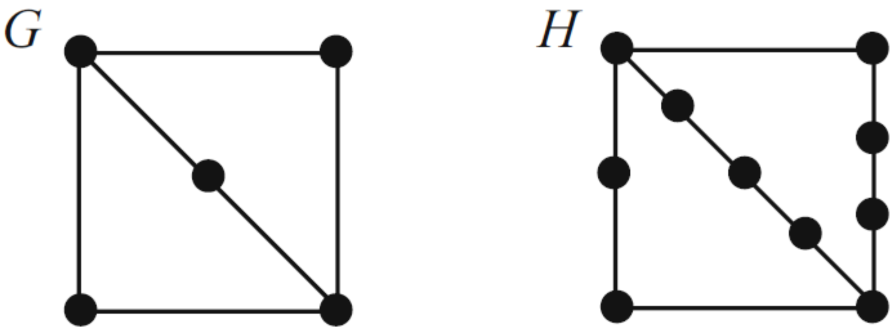


FIGURE 1.84. A graph and a subdivision.

Kuratowski's Theorem

- Theorem (1.39, H; Ex1, S1.5.4, H) A graph G is planar $\Leftrightarrow$ every subdivision of G is planar

- Theorem (1.40, H; Kuratowski 1930) A graph is planar $\Leftrightarrow$ it contains no subdivision of $K_{3,3}$ or K_5